SANDIA REPORT

SAND 2001-3513
Unlimited Release
Printed November 2001

A Hierarchical Bayes Approach to System Reliability Analysis

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Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550
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Abstract

The Comprehensive Test Ban Treaty of 1996 banned any future nuclear explosions or testing of nuclear weapons and created the CTBTO in Vienna to implement the treaty. The U.S. response to this was the cessation of all above and below ground nuclear testing. As such, all stockpile reliability assessments are now based on periodic testing of subsystems being stored in a wide variety of environments.

This data provides a wealth of information and feeds a growing web of deterministic, physics-based computer models for assessment of stockpile reliability. Unfortunately until 1996 it was difficult to relate the deterministic materials aging test data to component reliability. Since that time we have made great strides in mathematical techniques and computer tools that permit explicit relationships between materials degradation, e.g. corrosion, thermo-mechanical fatigue, and reliability. The resulting suite of tools is known as CRAX and the mathematical library supporting these tools is *Cassandra*.

However, these techniques ignore the historical data that is also available on similar systems in the nuclear stockpile, the DoD weapons complex and even in commercial applications. Traditional statistical techniques commonly used in classical reliability assessment do not permit data from these sources to be easily included in the overall assessment of system reliability. An older, alternative approach based on Bayesian probability theory permits the inclusion of data from all applicable sources. Data from a variety of sources is brought together in a logical fashion through the repeated application of inductive mathematics.

This research brings together existing mathematical methods, modifies and expands those techniques as required, permitting data from a wide variety of sources to be combined in a logical fashion to increase the confidence in the reliability assessment of the nuclear weapons stockpile.

The application of this research is limited to those systems composed of discrete components, e.g. those that can be characterized as operating or not operating. However, there is nothing unique about the underlying principles and the extension to continuous subsystem/systems is straightforward. The framework is also laid for the consideration of systems with multiple correlated failure modes. While an important consideration, time and resources limited the specific demonstration of these methods.

Contents

A Hierarchical Bayes Approach to System Reliability Analysis ..... *i*
Contents i
Figures iii
Tables iii

A Hierarchical Bayes Approach to System Reliability Analysis *1*

Abstract **1**

Background **1**
Problem 1
Solution Approach 2

Why Bayes Methods for Stockpile Assessment? **2**
Current Approach 3
Assumptions (see Figure 1) 3
Selection of Sample 4
Observation of Defects 4
Alternative approach 6
Assumptions 6
Observation of Defects 7
Summary 8

Alternative Bayesian Techniques **8**
Empirical Bayes 8
Example 9
Hierarchical Bayes 9
Example revisited 10
Graphical Representation of Hierarchical Models 11

Markov Chain Monte Carlo Methods **12**
Metropolis-Hastings 13
Gibbs Sampling 15
Full Conditional Distribution 17
Exchangeability 18

Bayesian System Reliability Analysis **18**
System Level Analysis 18
System Definition 19
Coherence 19
Reliability 19
Aggregation of System/Component Data 20
Caution Regarding Aggregation 20

Reliability Simulation Using MCMC **20**
Component Level Analysis 20
Prior distribution – Example 1 21
Prior distribution – Example 2 22
Component Data 22
Construction of Full Conditional Distributions (without system level test data) 22
Simulation 23
Example 24
Stationarity 26
System Level Analysis with Component Data 26
Independent Failure Modes 26
System and Component Data Aggregation 28

Example 29
Correlated System Failure Modes 31
Expert Elicitation to Construct Similarity Relationships 33
Discussion 33
Summary 33
References 34
Appendix A. Function Definitions 36
Appendix B. Monte Carlo Sampling 37
Monte Carlo Sampling 37
Random Number Generation 37
Inversion 37
Transformation 38
Acceptance-Rejection Sampling 39

Figures

Figure 1. Sample Parameters 3
Figure 2. Probability Density Functions of Defect Rates 6
Figure 3. Interference Example of Hierarchical Bayes 12
Figure 4.a-b Metropolis-Hastings Sample History 15
Figure 4.c Metropolis-Hastings Results 15
Figure 5.a-b Results for Example 1 of the Gibbs Sampler 17
Figure 6. Failure Data Processing Options 20
Figure 7.a Prior Family 21
Figure 7.b Hyperprior Family 21
Figure 8. DAG for Example 1 21
Figure 9. Component DAG for Example 2 22
Figure 10.a-b Results for Component Reliability Example 25
Figure 10.c-d Results for Component Reliability Example 25
Figure 11.a-b Results Using Extreme Initial Values 26
Figure 12. Simple System 26
Figure 13.a-b Results for Components 1 and 2 27
Figure 13.c-d Results for Components 3 and 4 28
Figure 14.a-b Results for System Level Analyses 28
Figure 15. Elements of Component Histogram 29
Figure 16. Typical Component Reliability Prior 29
Figure 17. Typical Parameter Prior Density Functions 30
Figure 18. Typical System Reliability 30
Figure 19. Typical Component Reliabilities – Case 6 31
Figure 19. Typical System Reliabilities – Case 5 and Case 6 31
Figure B.1. Rejection Sampling 39

Tables

Table 1 Summary of Results for a Random Sample of 11. 6
Table 2. . Summary of Results for a Targeted Samples 6
Table 3. Summary of Results for a Targeted Samples Using Alternative Methodology 8
Table 4. Empirical Bayes Example 9
Table 5. First and Second Stage Parameters 27
Table 6. Results of 9 Sample Cases 30

A Hierarchical Bayes Approach to System Reliability Analysis

Abstract

The Comprehensive Test Ban Treaty of 1996 banned any future nuclear explosions or testing of nuclear weapons and created the CTBTO in Vienna to implement the treaty. The U.S. response to this was the cessation of all above and below ground nuclear testing. As such, all stockpile reliability assessments are now based on periodic testing of subsystems being stored in a wide variety of environments.

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However, these techniques ignore the historical data that is also available on similar systems in the nuclear stockpile, the DoD weapons complex and even in commercial applications. Traditional statistical techniques commonly used in classical reliability assessment do not permit data from these sources to be easily included in the overall assessment of system reliability. An older, alternative approach based on Bayesian probability theory permits the inclusion of data from all applicable sources. Data from a variety of sources is brought together in a logical fashion through the repeated application of inductive mathematics.

This research brings together existing mathematical methods, modifies and expands those techniques as required, permitting data from a wide variety of sources to be combined in a logical fashion to increase the confidence in the reliability assessment of the nuclear weapons stockpile.

The application of this research is limited to those systems composed of discrete components, e.g. those that can be characterized as operating or not operating. However, there is nothing unique about the underlying principles and the extension to continuous subsystem/systems is straightforward. The framework is also laid for the consideration of systems with multiple correlated failure modes. While an important consideration, time and resources limited the specific demonstration of these methods.

Background

Problem

Current techniques and analysis tools used to assess stockpile reliability do not permit the mixture of data from system, subsystem, component, etc. level tests. The methods used depend exclusively on full scale testing for system reliability evaluation and ignore stockpile historical data, commercial product history and the judgment of engineering

designers. There is a need to be able to focus testing on those subsystems where there will be the largest return for testing dollars. In this case, return on investment is defined as an increased confidence in stockpile reliability. 'Focused' testing such as this requires a trade-off between the amounts of testing at the subsystem level versus testing at the system level. In addition, optimal development of test plans requires consideration of the time dependent nature of material properties (corrosion, thermo-mechanical fatigue, etc.), costs of manufacturing and testing unique subsystems or components, the historical data on similar systems (commercial and stockpile), and statistically dependent failure modes. Fundamental mathematical techniques exist in the open literature to address some of these issues and have been implemented in various software tools throughout Sandia. However, existing techniques and software tools do not address the critical issues of time dependent performance degradation and statistically dependent failure modes. Nor do they consider the optimal allocation of test resources. More detailed investigation of these methods will be necessary before a final conclusion can be reached.

Solution Approach

This research is focused on the use of Bayesian methods as the fundamental mathematical tool for addressing the problem raised above. Particular concerns in testing complex systems are the high costs of testing and making accurate predictions regarding performance through the maximum use of all available information. This information might include, for example, engineering experience on similar systems or experimental data.

To appreciate why a Bayesian approach has been taken, consider the problem of a bag containing 5 red balls and 7 green balls (Jaynes 1989). On a particular draw we choose a ball, with probability 5/12 and 7/12 of picking a red or a green ball respectively. If, after the initial selection, the ball is not returned to the bag, then the chances of picking either a green or red ball on the next selection depends on the prior selection. On the other hand, if no information regarding the result of the first selection is available and a green ball is chosen on the second draw, what can be said about the probability of choosing a red or green ball on the first pick? Intuition suggests the results of the second selection should not influence the probability of choosing a red or green ball on the first draw. However, before answering this, consider the situation where there are only 1 red and 1 green ball in the bag. Clearly, the information available as a result of the second draw influences the guess as to the first selection. It is this use of information in a conditional manner that provides additional insight into problems not otherwise possible and is the key to a Bayesian approach to test plan design and data analysis.

Why Bayes Methods for Stockpile Assessment?

From 1945 to 1963 the reliability of the U.S. nuclear stockpile was guaranteed through a series of atmospheric tests. After the 1963 Limited Test Ban treaty testing by the U.S., Russia and the United Kingdom was conducted underground. Finally, the Comprehensive Test Ban Treaty of 1996 banned any future nuclear explosions, testing of nuclear weapons, or any other nuclear explosions and created the CTBTO in Vienna to implement the treaty. As such, all stockpile reliability assessments are now based on assessment of subsystems being stored in a wide variety of environments.

However, in addition to these tests, as the stockpile is brought in for aging and surveillance inspection, data is collected on materials degradation, component performance drift, etc. . The limited test and inspection data that is available provides a wealth of information for assessment purposes. These test results feed a growing web of complex physics-based computer models for assessment of stockpile reliability. Until approximately 1996, it was difficult to relate the deterministic materials aging test data to component reliability. At that time efforts were initiated to develop a suite of mathematical techniques and computer tools that would permit an explicit relationship to be explored between materials degradation, e.g. corrosion, thermo-mechanical fatigue, and reliability. This suite of tools is referred to as CRAX and the mathematical library that supports these tools is *Cassandra*.

However, in addition to explicit experimental testing of material properties, an abundance of historical data is available on similar systems in the nuclear stockpile, the DoD weapons complex and even in commercial applications. Unfortunately traditional statistical techniques commonly used in classical reliability assessment do not permit data from these sources to be included in the overall assessment of system reliability. A Bayesian approach to reliability assessment permits the inclusion of data from all applicable sources. The data is brought together in a logical fashion through the repeated application of inductive mathematics.

To understand the usefulness of the Bayesian approach and the impact on the test programs being conducted, it is best to examine a very simple example that, in an abstract fashion, contrasts the current method of stockpile evaluation with a very simple Bayesian perspective.

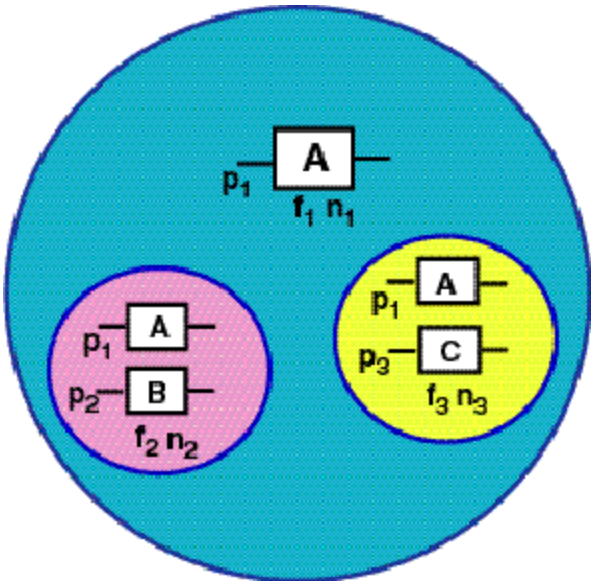

Figure 1. Sample Parameters

Current Approach

The assumptions used in the section are based on those outlined in the white paper entitled *Analysis of Stockpile Sampling Strategies*, by S. Crowder and E. Collins, 4 May 2000. The distribution of subpopulation fractions was adjusted slightly to assure that the fraction total was unity.

Assumptions (see Figure 1)

1. The general population has an overall defect rate of p_1. All of the defects associated with this rate will be lumped together and labeled as failure mode **A**. The fraction of the population with only failure mode A is f_1
2. A certain fraction of the population, f_2, has an additional inherent failure mode **B**, that occurs with rate p_2. This fraction of the population can experience both failure modes **A** and **B**.

3. A third fraction of the population, f_3, has an additional unknown failure mode **C**, that occurs with rate p_3. This fraction of the population can experience both failure modes **A** and **C**.
4. These three population fractions are mutually exclusive and are representative of the dominant population failure modes.
5. The failure modes are pair-wise independent.
6. The current approach to sampling requires a sample of 11 systems to be randomly selected from the population each year. No bias is introduced toward/away from selecting particular units; i.e. the sampling is not done in a stratified fashion.
7. The population fractions are: $f_1 = 0.80$, $f_2 = 0.10$, $f_3 = 0.10$ and the assumed defect rates are: $p_1 = p_2 = p_3 = 0.1$.

Selection of Sample

Given a sample of N units from the stockpile, there is a certain probability that the sample will be composed of n_1, n_2 and n_3 samples from the three fractions (where $N = n_1 + n_2 + n_3$). The probability mass function of choosing a particular sample combination can be described with a multinomial distribution:

$$P(X_1 = n_1, X_2 = n_2, X_3 = n_3) = \frac{N!}{n_1! n_2! n_3!} f_1^{n_1} f_2^{n_2} f_3^{n_3}. \quad (1)$$

For example, the probability of selecting a sample composed of 8 units from the first population, 2 units from the second and 1 unit from the third can be calculated:

$$P(X_1 = 8, X_2 = 2, X_3 = 1) = \frac{11!}{8!2!1!}(0.80)^8 (0.10)^2 (0.10)^1 = 0.083.$$

The probability of not sampling from the second subpopulation in the first year (of a two year sampling interval) can be calculated to be 0.318 and the probability of not sampling from the third subpopulation is also 0.318. The probability that the sample of 11 will not contain units from either subpopulation is 0.086. Note that these numbers only relate to the probability that the overall sample will contain at least one unit from the subpopulations. Even if the subpopulation is sampled, there is nonzero probability that no defects will be observed.

Observation of Defects

Given a sample of n_i units, the probability of observing s_i successes or equivalently $d_i = n_i - s_i$ defects is described by the binomial distribution:

$$P(S_i = s_i \mid p_i, N_i = n_i) = \frac{n_i!}{s_i!(n_i - s_i)!}(1 - p_i)^{s_i} (p_i)^{n_i - s_i} \quad (2)$$

for each subpopulation. To calculate the probability of observing a defect(s) in a sample of *N* units, it is necessary to use the Theorem of Total Probability and convolve Equations 1 and 2:

$$P(S_i = s_i) = \sum_{n_i=1}^{N} P(S_i = s_i \mid p_i, N_i = n_i) P(X_i = n_i) \quad (3)$$

where the marginal probability mass function is given by:

$$P(X_i = n_i) = \sum_{n_j=1}^{N} \sum_{n_k=1}^{N} P(X_i = n_i, X_j = n_j, X_k = n_k). \quad (4)$$

Care must be taken when calculating the probability of observing defects for each of the failure modes since the total number of samples is limited to a specified number, in this case 11. Once the number of samples from two subpopulations is chosen, n_i, n_j, the number of samples from the remaining subpopulation is determined: $n_k = N - (n_i + n_j)$.

Since there will always be *N=11* samples that can be used to detect failure mode **A**, the probability of observing s_i successes is governed entirely by Equation 2. The probability of not observing a failure in the general population that can be traced to the common failure mode **A** can be calculated to be 0.314 and the expected number of years until a defect of type **A** is observed is $1/(1-0.314) \approx 1.5$ years. This number is independent of whether targeted sampling is accomplished.

Utilizing Equation 3, the probability of not observing a defect with a root cause attributable to failure mode **B** during a random year of testing is 0.895. The probability of not observing failure mode **C** is also 0.895. The expected number of years before observing at least one defect of failure type **B** is therefore: $1/(1-0.895) \approx 10$ years. Similarly, the expected number of years before observing at least one defect related to failure type **C** is: $1/(1-0.895) \approx 10$ years.

Alternatively, suppose a sample of 4 units is selected from a specific subpopulation, e.g. the subpopulation associated with failure mode **B**. Then for that subpopulation, the probability of observing s_i successes is again governed entirely by Equation 2. The probability of not observing a defect in the general population that can be traced to the common failure mode **B** can be calculated to be 0.656 and the expected number of years until a defect is observed is $1/(1-0.656) \approx 3$ years.

Considering that there now remain at most only 7 samples to detect failures for mode **C**, the probability of detecting a defect associated with failure mode **C** can be calculated by combining Equations 2 and 3. The probability of not detecting a failure of type C is 0.932 and it is expected to take $1/(1-0.932) \approx 15$ years until a defect is observed.

Of course, a major fault with this analysis is that assumption that you would continue targeting a particular segment of the population even after not observing a particular failure mode. In reality, one would not continue sampling after there was sufficient confidence that the original hypothesis of a particular failure mode had been substantiated. The natural question then arises: how does one measure this degree of confidence regarding the significance of a particular failure mode?

Failure Mode	A	B	C

Prob{detect defect in 1 yr}	0.686	0.105	0.105
Expected time to detect defect (yrs)	1.5	10	10

Table 1. Summary of Results for a Random Sample of 11

Failure Mode	A	B	C
Prob{detect defect in 1 yr}	0.686	0.344	0.068
Expected time to detect defect (yrs)	1.5	3	15

Table 2. Summary of Results for a Targeted Samples of 7 (subpopulation 2) and 4 (subpopulations 1 &3)

Alternative approach

In the following, an alternative approach is presented which explicitly accounts for data that might be available from various other testing activities. While there is no substitute for full scale testing in a realistic operational environment, it is also difficult justify not considering data from all relevant sources.

Assumptions

The alternative approach is still consistent with assumptions depicted in Figure 1. In the traditional approach outlined above, it was assumed that a particular subpopulation, e.g. **B**, would be targeted for sampling without any prior knowledge that a potential problem exists. It was also assumed that this failure mode had an associated defect rate of p_2, even though there was no knowledge that this failure mode exists.

As with the traditional approach, it will be assumed that the number of success, *s*, observed in *n* trials is governed by the binomial probability mass function:

$$P(S = s \mid p, N = n) = f(s \mid p) = \frac{n!}{s!(n-s)!}(1- \quad (5)$$

However, in this alternative approach it will be assumed that the rate at which defects occur is a random variable, i.e. the true defect rate for each subpopulation is assumed to be unknown. Based on information available prior to the sample generation and testing, the probability density functions are assumed to be characterized by a beta density function:

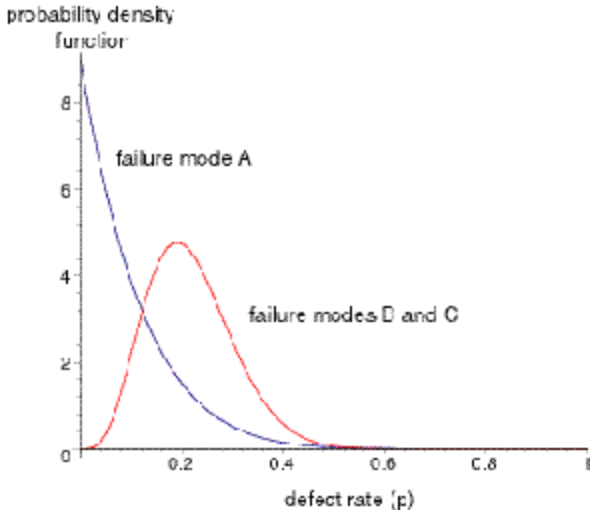

Figure 2. Probability Density Functions of Defect Rates

$$g(p) = \frac{\Gamma(n_0)}{\Gamma(s_0)\Gamma(n_0 - s_0)} p^{s_0 - 1}(1-p)^{n_0 - s_0 - 1} \quad 0 \le p \le 1 \tag{6}$$

where: $\Gamma(n) = \int_0^\infty x^{n-1}\exp(-x)dx$, and s_0, n_0 are parameters of the distribution.

After conducting a series of *n* tests and observing *s* successes, a new estimate of the defect rate can be calculated. Bayes' theorem provides a relationship between the new defect distribution (the posterior distribution) and the prior distribution (Equation 6):

$$g(p \mid s) = \frac{f(s \mid p)g(p)}{\int_0^1 f(s \mid p)g(p)dp} = \frac{f(s \mid p)g(p)}{f(s)} \tag{7}$$

Given the assumption of a general beta density function as the prior distribution on defect rate and a binomial distribution on the number of successes, the posterior distribution is also a beta distribution:

$$g(p \mid s) = \frac{\Gamma(n_0 + n)}{\Gamma(s + s_0)\Gamma(n_0 + n - s_0 - s)} p^{(s_0 + s) - 1}(1-p)^{(n_0 + n - s_0 - s) - 1} \quad 0 \le p \le 1 \tag{8}$$

Observation of Defects

The probability mass function of the number of successes in *n* trials (considering all possible values of the defect rate predicted by the posterior distribution) is the beta-binomial distribution:

$$\begin{aligned} f(s) &= \int_0^1 f(s \mid p)g(p)\,dp \\ &= \frac{n!\Gamma(n_0)\Gamma(s_0 + s)\Gamma(n_0 + n - s_0 - s)}{s!(n-s)!\Gamma(n_0 + n)\Gamma(s_0)\Gamma(n_0 - s_0)} \qquad 0 \le s \le n \end{aligned} \tag{9}$$

The function, *f(s)*, can be used to calculate the probability of observing one or more defects in a sample of *n* units given the current test data and prior information on the underlying defect rate.

For this example, it is assumed that during the surveillance program there were 23 tests conducted and 5 failures were observed related to failure mode B. No failures have yet been observed for failure mode C. These numbers can be modified to specifically address the testing that was accomplished for a particular component.

The estimate of the number of years to detect a problem is greatly simplified and is based on not updating the failure/success information that is gathered during testing each year.

Failure Mode	A	B	C
Prob{detect defect in 1 yr}	0.550	0.600	0.556
Expected time to detect defect (yrs)	1.82	1.67	1.80

Table 3. Summary of Results for Targeted Samples of 7 (subpopulation 2) and 4 (subpopulations 1 &3) Using Alternative Methodology

Summary

The use of condition-based logic – *given this information then I expect these results* – contrasts greatly with more popular approach – *this exact situation has happened many times so I expect it will happen again*, or equivalently – *this has never happened before so it will never happen in the future*. The application of Bayesian methods and inductive reasoning in general, permits the analyst to provide answers to a variety of questions with increased confidence.

The above very simple example demonstrated how, at no additional cost, existing data could be included in the reliability assessment and highlight a potential problem in a significantly shorter period of time.

The objective of this research is to explore the application of Bayesian methods in assessing stockpile reliability and the use of these techniques in developing test planning strategies that will provide the nuclear weapons community with increased confidence in the system reliability being reported.

Alternative Bayesian Techniques

Within the broad family of Bayesian analysis techniques there are two broad frameworks for integrating test data from various sources and at various system and subsystem levels: empirical Bayes and hierarchical Bayes. The terms 'empirical' and 'hierarchical' are unfortunately common in the literature; all Bayesian methods are empirical in nature and all can be described as being hierarchical in the fashion in which data is accumulated.

Hierarchical Bayes (HB) is the more recent addition to the family and is more efficient than the traditional empirical Bayesian approach and has been chosen as the general direction for future research. HB approach is also less sensitive to the choice of the prior distribution parameters, typically the source of much discussion. A very brief discussion of each method is provided in the following sections.

Empirical Bayes

The foundation for empirical Bayes, or more specifically, parametric empirical Bayes, has been in place since von Mises in the 1940's, but really came into prominence in the 1970's with the series of papers by Efron and Morris, (e.g. Efron and Morris 1972). There have been a number of excellent publications in which the authors have taken the effort to explain the theory and logic behind empirical Bayes (Casella 1985, Deeley and Lindley 1981, Kass and Steffey 1989 and Morris 1983) and its relationship to other statistical techniques.

The following discussion draws heavily from the example presented in Gelman, et. al. (2000, p. 120). The example has been modified slightly to provide some intuition to stockpile reliability evaluation.

Example

Assume that success/test data from one particular weapon system is available from previous test cycles is summarized in Table 4.

The objective of the effort is to estimate the probability of failure or defect rate, *p*. It will be assumed that the number of failures in each of the 10 tests (9 previous, 1 current), will be binomial distributed random variable with a given a defect rate *p*. The population from which the samples for the ten tests were drawn will be assumed to be absolutely homogeneous.

Further, we will assume that *p* has a prior distribution described by a Beta probability density function with known parameters α and β: B(α, β). The posterior distribution of p after test 10 is also a Beta distribution: B(α+1, β+4). Using the traditional statistical approach of method of moments where:

$$a + b = \frac{E(p)[1 - E(p)]}{V(p)} - 1,$$
$$a = (a + b)E(p)$$

these parameters can be estimated from the data: α = 0.6, β = 3.13. The posterior distribution is therefore B(1.6, 7.13) with a mean and variance of 0.183 and 0.015, respectively. Compared to the estimate of *p* from the current data, 0.25, the posterior estimate of *p* is significantly smaller (p=0.183).

Some subtle problems exist with the above approach. First, the use of point estimates for α and β are arbitrary and lack consideration for modeling uncertainty. Second, if there was interest in performing individual analyses on the 9 previous tests, the data would have to used once to form a prior and then again as part of the posterior. Data would be used twice and result in an overly conservative estimate of the defect rate *p*.

In general empirical Bayesian methods represent only an approximation to a full Bayesian analysis. It does not represent a true Bayesian analysis of the data since a traditional statistical approach was used to estimate the parameters of the prior distribution. Alternatively, in a hierarchical Bayes approach, data analysis, all prior and posterior distribution characteristics are estimated in an integrated fashion.

Test	Units tested - n_j	Failures- y_j	P estimate p_j
1	5	1	0.2
2	4	0	0.0
3	5	0	0.0
4	6	1	0.17
5	4	1	0.25
6	4	2	0.5
7	5	0	0.0
8	6	0	0.0
9	6	2	0.33
Total	45	7	
10	5	1	0.25

Table 4. Empirical Bayes Example

Hierarchical Bayes

As the section title suggests, the distinguishing feature of the alterative approach to empirical Bayesian analysis is the hierarchical nature in which information is accumulated. Define

$\mathbf{y} = \{y_1, \ldots, y_N\}$ to be a set of independent and identically distributed samples and the associated likelihood function is therefore $l(\mathrm{q}, \mathbf{p} \mid \mathbf{y})$. Using the example from the previous section, it will be assumed that the variable $\mathrm{q} = \{a, b\}$ is an unknown hyperparameter vector described via a prior distribution $f(\mathrm{q})$. In the previous example, it was assumed that $\mathrm{q} = \{a, b\}$ was known and only required estimation. Now it is assumed that it is a random variable and the uncertainty in the hyperparameters is addressed explicitly. (The variable *p* will be generalized as a vector quantity, **p**, to allow for multiple distribution parameters.) The complete Bayesian analysis for characterizing the density function $f(\mathbf{p} \mid \mathbf{y})$ requires a description of the vector of random variables $(\mathbf{p}, \mathrm{q})$ with joint prior distribution: $f(\mathbf{p}, \mathrm{q}) = f(\mathbf{p} \mid \mathrm{q}) f(\mathrm{q})$ and joint posterior distribution:

$$\begin{aligned} f(\mathbf{p}, \mathrm{q} \mid \mathbf{y}) &\propto f(\mathbf{p}, \mathrm{q}) l(\mathbf{p}, \mathrm{q} \mid \mathbf{y}) \\ &= f(\mathbf{p}, \mathrm{q}) f(\mathbf{y} \mid \mathrm{q}) \end{aligned}$$

Note that the joint posterior density function: $f(\mathbf{p}, \mathrm{q} \mid \mathbf{y})$ can be written as a product of the hyperprior $f(\mathrm{q})$, the population distribution $f(\mathbf{p} \mid \mathrm{q})$ and the likelihood function $l(\mathrm{q}, \mathbf{p} \mid \mathbf{y})$. Under the assumption of an independent and identically distributed set of samples, $\mathbf{y} = \{y_1, \ldots, y_N\}$, an analytical expression for the conditional posterior density of $f(\mathbf{p} \mid \mathrm{q}, \mathbf{y})$ can be easily constructed as the product of the density functions $f(y_i)$. Note also that $f(\mathbf{y} \mid \mathbf{p}, \mathrm{q}) = f(\mathbf{y} \mid \mathrm{q})$ since **y** does not directly depend on **p** since $\mathrm{q}(\mathbf{p})$.

In the case of conjugate density functions a solution is available directly. Once an expression for the joint posterior function is found, the marginal posterior distribution can be found through direct evaluation or via integration:

$$f(\mathbf{p} \mid \mathbf{y}) = \frac{f(\mathbf{p}, \mathrm{q} \mid \mathbf{y})}{f(\mathrm{q} \mid \mathbf{p}, \mathbf{y})} = \int f(\mathbf{p}, \mathrm{q} \mid \mathbf{y}) d\mathrm{q} .$$

Example revisited

We will again assume that the number of failures in each test is an independent observation from a Binomial distribution: $y_j \approx \mathrm{Bin}(n_j, p_j)$. In addition, the parameter *p* will be assumed to be a random variable following a Beta distribution: $p_j \approx \mathrm{B}(a, b)$.

The joint posterior distribution is defined:

$$\begin{aligned} f(p, a, b \mid \mathbf{y}) &\propto f(a, b) f(p \mid a, b) l(\mathbf{y} \mid a, b, p) \\ &\propto f(a, b) \prod_{j=1}^{N} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p_j^{a-1} (1-p_j)^{b-1} \prod_{j=1}^{N} p_j^{y_j} (1-p_j)^{n_j - y_j} \end{aligned}$$

The marginal distributions of the parameters can then be found using the conditional probability expression, $f(p \mid \mathbf{y}) = \frac{f(p, a, b \mid \mathbf{y})}{f(p \mid a, b, \mathbf{y})}$:

$$f(p \mid a, b, \mathbf{y}) = \prod_{j=1}^{N} \frac{\Gamma(a+b+n_j)}{\Gamma(a+y_j)\Gamma(b+n_j-y_j)} p_j^{a+y_j-1} (1-p_j)^{b+n_j-y_j-1}$$

$$f(a,b \mid \mathbf{y}) \propto f(a,b) \prod_{j=1}^{N} \frac{\Gamma(a+b)\Gamma(a+y_j)\Gamma(b+n_j-y_j)}{\Gamma(a)\Gamma(b)\Gamma(a+b+n_j)}$$

To proceed, it is only necessary to describe a hyperprior distribution function for $\mathrm{q} = \{a, b\}$. Typically this distribution is taken to be relatively non-informative so avoid dominating the solution. Regardless of the choice of hyperprior, it is critical that it be mathematically feasible and results in a logical marginal posterior distribution.

An alternative to the analytical approach outlined above is of simulation to construct the various conditional density functions.

1. Generate a sample of the hyperparameter vector q from the marginal distribution function, $f(\mathrm{q} \mid \mathbf{y})$.
2. Given q, generate a sample parameter vector $\mathbf{p}$ from $f(\mathbf{p} \mid \mathrm{q}, \mathbf{y})$.
3. A population sample can be then be generated using the likelihood function $l(\mathbf{y} \mid \mathrm{q}, \mathbf{p})$.

Generally these steps will be difficult to accomplish due to the problems associated with generating samples from complex conditional distributions. A simulation technique particularly suited for this task, Markov Chain Monte Carlo simulation, will be introduced in the following section.

Two recent applications of hierarchical Bayes methods in the structural reliability area are Wilson and Taylor (1997) and Celeux et al. (1999).

Graphical Representation of Hierarchical Models

Hierarchical models have been increasingly popular and have a potential application for solving very complex problems. The structure of hierarchical models lends itself easily to a graphical depiction of the relationships between various model constructs using directed graphs. These graphical cartoons are useful for organizing information and also for constructing the posterior distribution functions discussed above.

Directed graphs are essentially a set of nodes connected with a set of directed edges or arrows which depict the informational dependencies between the nodes. Those nodes that feed information to subsequent nodes are considered *parent* nodes. Each node is considered independent of all other nodes except parent nodes and those nodes for which that node is a parent.

There are three types of nodes:

1. Constant nodes have no parents and represent fixed quantities in the analysis. These nodes are represented by rectangles.
2. Stochastic nodes may have parents or children and represent, typically, unobserved random variables. They are commonly represented by circles on the graph.
3. Functional or deterministic nodes represent functions of other variables in the graph.

Conventional depiction of the edges includes the use of solid arrows for statistical dependence and dashed lines for functional dependence. Note that even though nodes may have no common ancestor, and thus are marginally independent, this independence may disappear when other quantities enter into the conditioning structure. Dependencies are often induced between variables as a result of observational data. A box surrounding a set of variables indicates that those variables are conditionally statistically independent given their parents.

In the classic example of stress-strength interference, $g = R - S$, where: $f_R(r \mid m_r, s_r) \Rightarrow N(m_r, s_r)$, $f_s(s \mid m_s, s_s) \Rightarrow N(m_s, s_s)$ and, for example: $f_{m_s}(m_s \mid a_s, b_s) \Rightarrow N(a_s, b_s)$, $f_{s_s}(s_s \mid g_s, t_s) \Rightarrow \text{Gam}(g_s, t_s)$ and so on. As indicated by the box in Figure 3, the variables R and S are assumed to be independent random variables in this example.

With the exception of special simple cases involving conjugate distributions, the generation of the above mentioned conditional density functions can be extremely difficult and in some situations impossible. An increasingly useful approach for generating these distributions is based on assuming that the desired joint posterior distribution is the stationary distribution that results from a Markov chain. The use of Monte Carlo simulation methods to generate this distribution is referred to as Markov Chain Monte Carlo (MCMC) simulation.

Markov Chain Monte Carlo Methods

Fundamental to all Markov Chain Monte Carlo (MCMC) samplers is the ability to generate random variables from a distribution indirectly without having to calculate the density explicitly. As the name implies, these simulation methods are based on the basic principles of Markov chains.

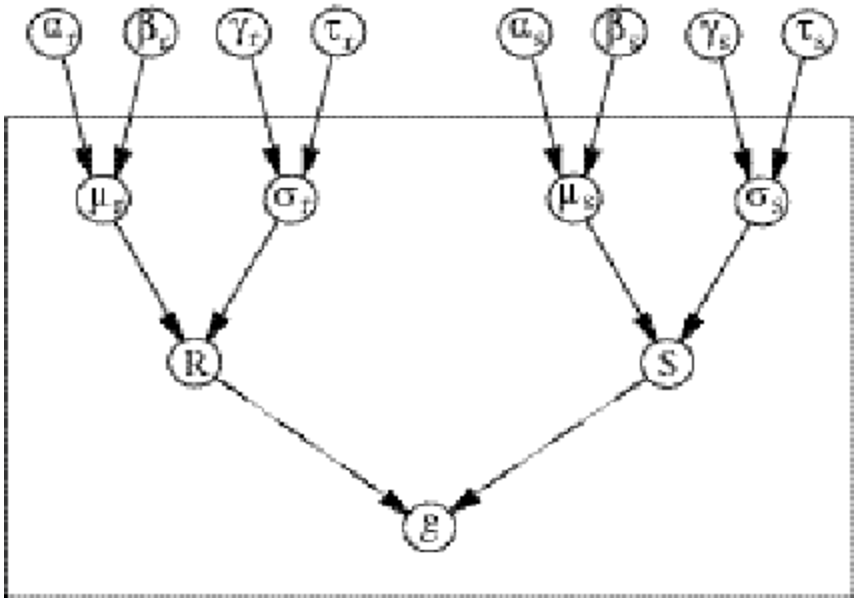

Figure 3. Interference Example of Hierarchical Bayes

Let $P(x,y)$ represent the transition matrix for a Markov Chain where $P(x,y)$ represents the probability of transitioning to a state *y* given that the current state is *x*. (Note that $\int_y P(x,y)=1$ and $P(x,x)$ may not be zero.) Assuming certain regularity conditions for $P(x,y)$, then after a sufficient number of transitions the probability of being in any particular state converges to a stationary probability density function $p(\mathbf{x})$. In particular, if $P(x,y)$ satisfies the reversibility criteria:$p(x)P(x,y)=p(y)P(y,x)$ then $p(\mathbf{x})$ is a stationary distribution.

For MCMC simulation, the problem is just the reverse: given a stationary probability density function $p(\mathbf{x})$, what is the necessary transition matrix? A method to accomplish this was first proposed by Metropolis (1953) and generalized by Hastings (1970).

Metropolis-Hastings

Assume that, as with the acceptance/rejection method above, a candidate density function, $q(x,y)$, exists for generating possible samples from $p(\mathbf{x})$. (In the MCMC literature $q(x,y)$ is referred to as the *proposal* distribution.) However, since samples are being generated from a Markov chain, this density function changes during the simulation process (depending on the current state). The density $q(x,y)$ can be easily constructed so that $\int_y q(x,y)dy=1$, however it may not satisfy the reversibility criteria.

The construction of an appropriate state transition matrix $P(x,y)$, for $y \neq x$, must account for the degree to which the reversibility criteria is satisfied. Define the probability of moving out of state *x* into any other state *y* as $a(x,y)$. Therefore the matrix describing transitions from x to y must be conditioned on this probability so that $P(x,y)=q(y,x)a(x,y)$. It can be shown (e.g. Chib and Greenberg 1995) that

$$a(x,y)=\min\left[\frac{p(y)q(y,x)}{p(x)q(x,y)},1\right], \quad \text{if } p(x)q(x,y)>0$$
$$=1, \quad \text{otherwise}$$

The value of $a(x,y)$ represents the probably that the *i+1* sample is a new sample, $x_{i+1}=y$, or is identical to the existing sample $x_{i+1}=x_i$. Contrast this with the acceptance/rejection method outlined above where, upon rejection, the sample is not retained.

The sampling method as first proposed by Metropolis assumes that the proposal density function is symmetric, i.e. $q(x,y)=q(y,x)$, which simplifies the criteria for accepting or rejecting the next sample: $a(x,y)=\min\left[\frac{p(y)}{p(x)},1\right]$.

For the situation of a scalar *x* the Metropolis-Hastings algorithm can be summarized:

1. Assume an initial value of $x_{(0)}$
2. generate a random deviate, *y* (the next potential candidate x value), from $q(x_{(i)},\cdot)$

3. sample $u = U_{0,1}$
4. If $u \le a(x_{(i)}, y) = \min\left[\frac{p(y)q(y, x_{(i)})}{p(x_{(i)})q(x_{(i)}, y)}, 1\right] \Rightarrow x_{(i+1)} = y$ else $x_{(i+1)} = x_{(i)}$
5. repeat steps 2-4 until a stationary sequence of variables of sufficient length has been generated

The proposal distribution can be of any form and a stationary distribution $p(\mathbf{x})$ will be generated. However, for faster convergence and better mixing, it is generally desirable that the proposal distribution be similar to the desired stationary distribution.

For a vector valued x, it is often desirable to sequentially update the individual components of the vector. Define $\mathbf{x}_{(i)} = \{x_{(i)1}, x_{(i)2}, \ldots, x_{(i)n}\}$ to be the ith n-dimensional sample vector generated and let $\mathbf{x}_{(i)-j} = \{x_{(i)1}, x_{(i)2}, \ldots, x_{(i)j-1}, x_{(i)j+1}, \ldots, x_{(i)n}\}$ be the same vector, but with the jth element removed. The Metropolis-Hastings algorithm then can be described:

1. Assume an initial value of $\mathbf{x}_{(0)} = \{x_{(0)1}, x_{(0)2}, \ldots, x_{(0)n}\}$
2. Generate a random deviate, $y_{(i)}$, from the proposal distribution $q(y_{(i)} \mid x_{(i)j}\mathbf{x}_{(i)-j})$. The value of $y_{(i)}$ represents the updated value of the jth component of $\mathbf{x}_{(i)}$ where the other elements have been updated in a previous iteration.
3. Sample $u = U_{0,1}$
4. If $u \le a(x_{(i)j}, \mathbf{x}_{(i)-j}, y_{(i)}) = \min\left[\frac{p(y_{(i)} \mid \mathbf{x}_{(i)-j})q(y_{(i)} \mid x_{(i)j}, \mathbf{x}_{(i)-j})}{p(x_{(i)j} \mid \mathbf{x}_{(i)-j})q(x_{(i)j} \mid y_{(i)}, \mathbf{x}_{(i)-j})}, 1\right] \Rightarrow x_{(i+1)j} = y_{(i)}$ else $x_{(i+1)j} = x_{(i)j}$. The other elements of $\mathbf{x}_{(i)}$ are not updated.
5. Steps 2-4 are repeated until all elements of $\mathbf{x}_{(i+1)}$ have been updated.
6. Steps 2-5 are repeated until a stationary sequence of samples $\mathbf{x}_{(i)}$ is available

It is very important to note that the distribution $p(\mathbf{x})$ needs to be only known up to a multiplicative constant, since this constant cancels out in the analysis. This can be critical in those situations in Bayesian reliability analysis where the constant can be difficult to calculate.

The availability of the full conditional distribution $p(x_{(i)j} \mid \mathbf{x}_{(i)-j}) = \frac{p(\mathbf{x})}{\int p(\mathbf{x})dx_{(i)j}} \propto p(\mathbf{x})$ is critical to MCMC analysis and is the subject of much of the literature. Common approaches will be outlined in the following discussion.

Example

Let $p(\mathbf{x}) = \frac{l}{x!} e^{-l}$ and the proposal distribution be a simple random walk:

$$y = \begin{cases} x_{(i)} + 1 \\ x_{(i)} - 1 \end{cases} \quad \text{with equal probability}$$

.

Since $q(x,y) = q(y,x)$ for a random walk process, the dependence on q drops out. The Metropolis acceptance ratio is therefore:

$$a = \frac{p(y)}{p(x_{(i)})} = \frac{x_{(i)}!}{y!} l^{\,y - x_{(i)}}$$

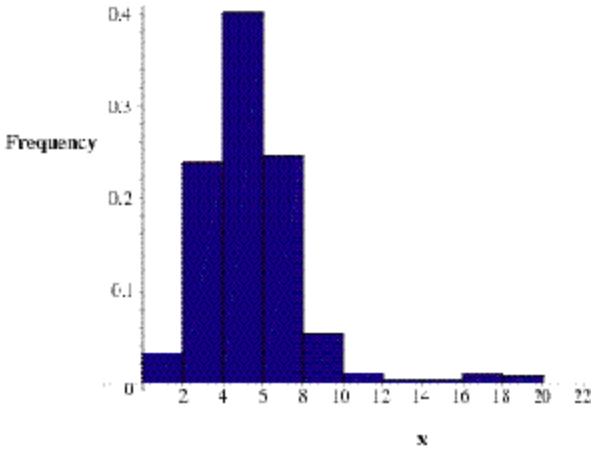

Figure 4.c Metropolis-Hastings Results

If $u = U_{0,1}$, then the transition to a new state will be accepted if $u < a$ and will be rejected otherwise. Figures 4.a,b,c illustrate the results for $l = 4.0$, $x_{(0)} = 20$ and a sample size of N=1000. Figure 4.a is of the first 100 samples and Figure 4.b is the complete history of 1000. Figure 4.c is a histogram depicting the resulting frequency distribution for the variable x.

Gibbs Sampling

Gibbs sampling is a special case of the Metropolis-Hastings sampling algorithm. The proposal distribution is assumed to be:

$$q\left(y_{(i)} \mid x_{(i)j}\mathbf{x}_{(i)-j}\right) = p\left(y_{(i)} \mid \mathbf{x}_{(i)-j}\right)$$

When this distribution is incorporated in the Metropolis-Hastings algorithm, the probability of accepting a transition is one. Gibbs sampling is identical to the iterative

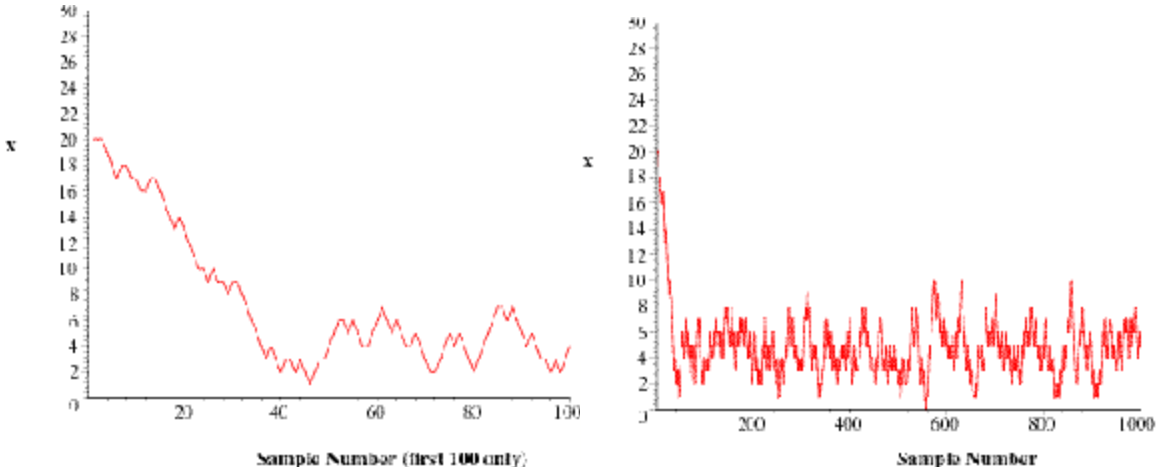

Figure 4.a-b Metropolis-Hastings Sample History

algorithm outlined in the previous section with no check for sample acceptance. Using the algorithm outlined by Gelfand and Smith (1990), a sample from the probability density function, $f(\mathbf{x}) = f(x_1, x_2)$, can be generated by iteratively samping from the full conditional distributions $f(x_1 \mid x_2)$ and $f(x_2 \mid x_1)$:

1. Assume an initial value of $\mathbf{x}_{(0)} = \{x_{(0)1}, x_{(0)2}, \ldots, x_{(0)n}\}$
2. Generate a random deviate, $y_{(i)}$, from the proposal distribution $p\left(y_{(i)} \mid \mathbf{x}_{(i)-j}\right)$. The value of $y_{(i)}$ represents the updated value of the j[th] component of $\mathbf{x}_{(i)}$ where the other elements have been updated in a previous iteration.
3. Step 2 is repeated until all elements of $\mathbf{x}_{(i+1)}$ have been updated.
4. Steps 2-3 are repeated until a stationary sequence of samples $\mathbf{x}_{(i)}$ is available. Once stationarity is achieved the resulting samples represent samples from $f(\mathbf{x})$.

The result of Steps 2-3 are realizations from a Markov chain with transition probability from $\mathbf{x}_{(i)}$ to $\mathbf{x}_{(i+1)}$ given by:

$$P\left(\mathbf{x}_{(i)}, \mathbf{x}_{(i+1)}\right) = \prod_{l=1}^{n} p\left(\mathbf{x}_{(i+1)l} \mid \mathbf{x}_{(i)j},\ j > l,\ \mathbf{x}_{(i+1)j},\ j < l\right)$$

Only the full conditional distributions $p\left(y_{(i)} \mid \mathbf{x}_{(i)-j}\right)$ are needed for Gibbs sampling. For high dimensional x, Gibbs sampling can be slow to converge. For this reason, if higher order conditionals are available, e.g. $f(x_2, x_3 \mid x_1)$ versus $f(x_3 \mid x_1, x_2)$, $f(x_2 \mid x_1, x_3)$, then the use of those conditionals can significantly speed convergence (Gelfland and Smith, 1990).

It should also be emphasized that variance reduction methods for generating samples (Latin hypercube, importance sampling, field analysis, etc.) are still applicable.

Example 1 (following Casella and George, 1992):

Let the number of failures observed on a system be a random variable described by a binomial distribution:

$$f(x \mid p, n) = \binom{n}{p} p^x (1-p)^{n-x}$$

where *p* is the probability of observing a defect in a particular trial. Given an observed number of failures, let *p* in turn be a random variable described by a Beta distribution:

$$f(p \mid x, n, a, b) = \frac{\Gamma(n+a+b)}{\Gamma(x+a)\Gamma(n-x+b)} p^{(x+a)-1}(1-p)^{(n-x+b)-1}$$

where α and β are referred to as the pseudo-number of failures and successes respectively. It is desired to obtain the distribution of the number of defects, $f(x)$, i.e. the marginal distribution.

Figure 5.a depicts the sampling history for variable x after 700 iterations with a Gibbs sampler using $a = 2$, $b = 4$ and n=16. Figure 5.b is the associated histogram. Since there is a closed form solution available for $f(x)$:

$$f(x) = \binom{n}{x} \frac{\Gamma(a+b)\Gamma(x+a)\Gamma(n-x+b)}{\Gamma(a)\Gamma(b)\Gamma(n+a+b)} \quad x = 0,1,\ldots n$$

the exact solution is superimposed on the histogram.

It is important to note that in those cases where the full conditional distributions are not available in closed form for easy sample generation, the analyst must resort to a Metropolis-Hasting algorithm.

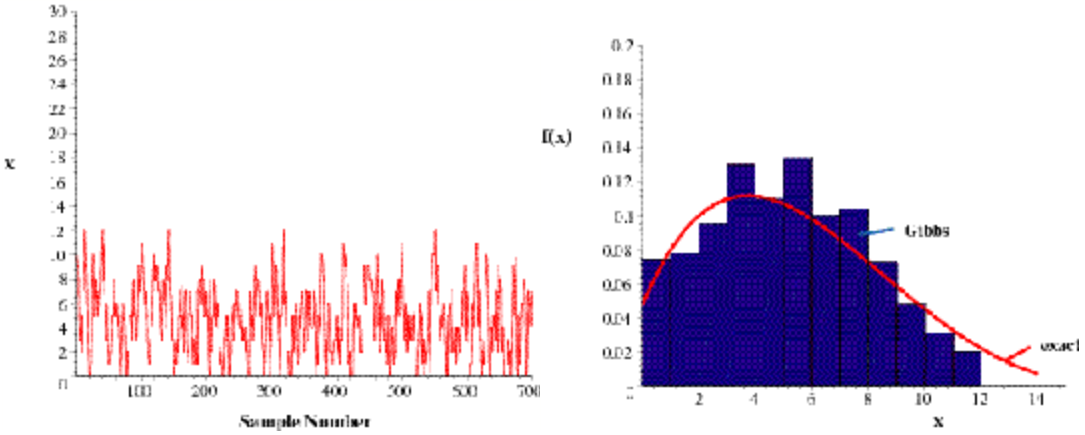

Figure 5.a-b Results for Example 1 of the Gibbs Sampler

Full Conditional Distribution

As noted previously, for Gibbs sampling, only the full conditional distributions are required. By specifying the relationships between these conditional distributions it is possible to completely describe the entire model. These relationships are fully characterized by the directed acyclic graph (DAG) discussed in the hierarchical section above. The DAG represents the conditional dependencies of the full model before any data is collected. When data is introduced to the model, the statistical relationships between variables may change.

Let **v** be the set of nodes on a DAG of interest and v_i be a particular node. Define $\mathbf{v}_{-i}$ to be set of nodes without the node v_i. A parent of v_i is any node with a directed line from it to node v. Lauritzen, et al. (1990) showed that a DAG model constructed in the above manner fully describes the joint distribution of all the random variables in terms of the conditional distribution of each node: $f(\mathbf{v}) = \prod P(v_i \mid \text{parents } [v_i])$. The full conditional distribution is therefore

$$f\left(x_j \mid \mathbf{x}_{-j}\right) = \frac{f(\mathbf{x})}{\int f(\mathbf{x})\,dx_j}$$
$$\propto f(\mathbf{x}) = f\left(x_j, \mathbf{x}_{-j}\right)$$
$$\propto \text{terms in } f(\mathbf{x}) \text{ involving } x_j$$
$$= \underbrace{P(x_j \mid \text{parents}\,[x_j])}_{\text{prior}} \underbrace{\prod_{x_j \in \text{parents}\,[w_k]} P(w_k \mid \text{parents}\,[w_k])}_{\text{likelihood}}$$

If full conditional distributions are difficult to sample from, an adaptive rejection method can be used (Gilks , 1992).

Example

Using the stress-strength example depicted in the DAG above, the joint distribution of all the parameters is given by:

$$f(R,S,m_r,s_r,a_r,b_r,g_r,t_r,m_s,s_s,a_s,b_s,g_s,t_s) =$$
$$f(R \mid m_r,s_r) f(m_r \mid a_r,b_r) f(s_r \mid g_r,t_r) f(S \mid m_s,s_s) f(m_s \mid a_s,b_s) f(s_s \mid g_s,t_s)$$

Exchangeability

Assume that over a period of time, a series of *N* binary (go/no-go) tests were performed. For the j^{th} test, a sample of n_j was evaluated and q_j failures were observed. If there is no information to distinguish any of the tests, there were no grouping or unique ordering, then there is symmetry in the parameters. This symmetry implies that it is possible to analyze the data regardless of the order in which the data was observed and processed, i.e. it is equally realistic to analyze the data as $(q_1, q_2, q_3, \ldots, q_n)$ or as $(q_n, q_2, q_1, \ldots, q_3)$. In this situation, where all the parameters are assumed to come from identical distributions, it is appropriate to use an *exchangeable* model. When additional data becomes available to support differences, then exchangeability may still be appropriate. For example, if after a series of tests, system age is identified as a distinguishing feature, this feature may be explicitly included in the model. In this way the dependencies may be encoded or grouped in such a manner that the resulting model is then exchangeable (Gelman, et al. 2000, p. 123-126).

Bayesian System Reliability Analysis

System Level Analysis

Over the past 35 years there has been considerable effort expended in characterizing the reliability of a complex system using Bayesian methods. The reader is referred to the many fine articles in the literature as well as the many books such as the classic by Martz and Waller (1982) and the more recent Savchuk and Tsokos (1996).

The majority of the literature concentrates on the development of approximations to unique system configurations and failure density functions such that the results are mathematically tractable. However, it is felt that the current reliability analysis problems to be addressed with this research can be better met with advanced computer simulation

methods. This approach requires a minimum of compromise in utilizing all available component and system operational information. In addition, prior distribution functions and system configurations are not 'forced' onto the analysis.

The material in this section is presented more to introduce notation necessary for future discussions rather than for completeness. The emphasis is on developing the mathematical foundation needed for combining system and component test information.

System Definition

It will be assumed that each of the $i, i = 1,\ldots,n$, components or subsystems, as well as the entire system can be in one of two possible states: operational or failed. [Note that all of the methods to be discussed can be generalized to multi-state system operation, and the possible dependence on time is understood.] The vector $\mathbf{x} = \{x_1,\ldots,x_n\}$ and the variable y completely represent the state of the subsystem and components respectively, where:

$$y, x_i = \begin{cases} 1 & \text{if (sub)system operating} \\ 0 & \text{if (sub)system failed} \end{cases} \qquad i = 1,\ldots,n$$

The system state is related to the subsystem states through a function $f(\cdot)$ referred to as the system structure function: $y = f(x_1,\ldots,x_n)$.

Coherence

Consider two n-tuples: $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$. The dominance of a over b is denoted: a $\geq$ b and is defined: a $\geq$ b iff $a_i \geq b_i$, $i = 1,\ldots,n$. Equivalence of a and b is denoted: a $=$ b and defined: a $=$ b iff $a_i = b_i$, $i = 1,\ldots,n$. A structure function, $f(\mathbf{x})$ is said to be coherent if: $\mathbf{x}' \geq \mathbf{x}'' \Rightarrow f(\mathbf{x}') \geq f(\mathbf{x}'')$ for all vectors $\mathbf{x}'$ and $\mathbf{x}''$.

Reliability

Let p_i, the probability that subsystem *i* is functioning, be a random variable with distribution function $p(p_i)$ defined over the unit interval. The expected reliability of the i^{th} component is therefore: $\hat{p}_i = E[p_i] = \int p_i p(p_i) dp_i$. It will be assumed that, given p_i, x_i is independent of $p_j, \forall_{j \neq i}$ and, given $\mathbf{p} = (p_1,\ldots,p_n)$, the x_i are all mutually independent.

For a series system, the reliability is given by:

$$p_S = \int \Pr\{y = 1 \mid p_1, p_2,\ldots,p_n\} \pi(p_1, p_2,\ldots,p_n) d\mathbf{p}$$

$$= \int \left(\prod_n p_i\right) \pi(p_1, p_2,\ldots,p_n) d\mathbf{p}$$

where $\pi(p_1, p_2,\ldots,p_n)$ is not necessarily equal to $\pi(p_1)\pi(p_2)\cdots\pi(p_n)$. Finally, a statement regarding the expected reliability of the system can be constructed:
$\hat{p}_s = E[p_s] = \int f(\mathbf{p}) p(\mathbf{p}) d\mathbf{p}$. Except for unique situations, evaluation of $\hat{p}_s$ is difficult to compute and various uncertainty analysis methods, e.g. Monte Carlo, must be applied.

Aggregation of System/Component Data

A number of authors have suggested methods for combining subsystem and system data under a Bayesian framework, including: Mastran (1976), Mastran and Singpurwalla (1978), Martz, Waller, and Fickas (1988) and Martz and Waller (1990). In general, these papers, along with a number of others, support analysis approaches that permit the inclusion of test data from a number of levels of system indenture, into an overall estimate of the system reliability.

The methods all depend on a combination of analytical techniques for combining test information and inherently depend on assumptions regarding the underlying distribution function. The approach presented in this paper is far less elegant, but applicable to wider array of complex system configurations.

Caution Regarding Aggregation

As noted in the papers above, the aggregation of system and component level data can become involved. In general, the system level reliability distribution derived from component data is used as a prior the reliability distribution based on system level data (Figure 6). However, when component and system level failure and performance information is collected at the same time, aggregation of component data into a system level analysis may not result in the same reliability prediction as obtained from the system data alone, e.g. Azaiez and Bier (1995).

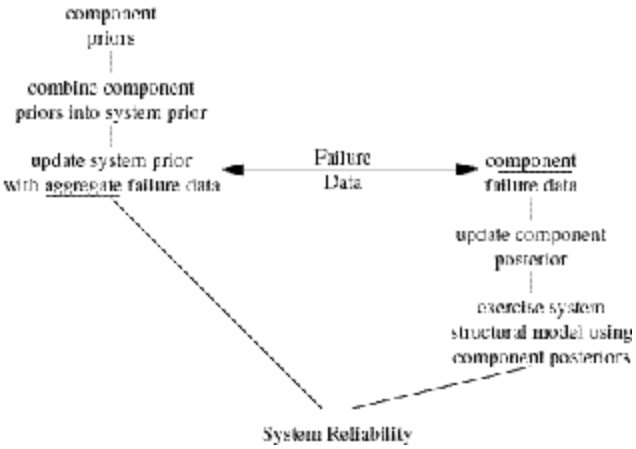

Figure 6. Failure Data Processing Options

Reliability Simulation Using MCMC

The preceding discussion has laid out the pieces that comprise a foundation for conducting a Markov Chain Monte Carlo analysis of a complex system. The following discussion focuses on the application of these techniques in the reliability assessment of such a system. The discussion begins with the development of two methods for performing component level analysis. Also addressed is the issue associated with the inclusion of experimental and historical data in the component reliability assessment. Finally, an approach for combining component level information into a system level reliability evaluation is presented with a suitable eye on the impact of aggregation of data.

Component Level Analysis

For now, assume that each component can be tested and results in the observation of either a success or failure. In this binomial sampling there are n tests performed, s successes observed and p is defined to be the probability of success on a single trial. The

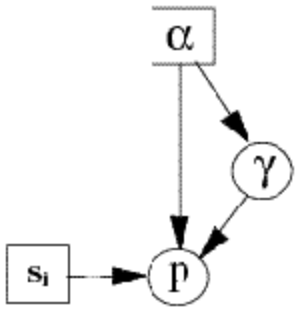

Figure 8. DAG for Example 1

conditional distribution of the number of failures observed given a particular probability of success is therefore:

$$P(S = s \mid p, N = n) = f(s \mid p) = \frac{n!}{s!(n-s)!} p^s (1-p)^{n-s} \qquad p > 0$$

Let $g(p)$ be the prior density function describing the probability of failure. The posterior distribution function is then given by:

$$g(p \mid s) = \frac{f(s \mid p) g(p)}{\int f(s \mid p) g(p) dp} = K p^s (1-p)^{n-s} g(p)$$

Prior distribution – Example 1

Each component will be assumed to be highly reliable and, following Chen and Singpurwalla (1996), the prior distribution on the probability of failure will be assumed to be a truncated beta distribution: Beta(1,α), defined over the interval $(g,1)$ (Figure 9.a):

$$p(p \mid g, a) = \frac{1}{B(a,b)} p^{a-1} (1-p)^{b-1} \quad 0 < p < 1, \quad a, b > 0$$
$$= a(1-p)^{a-1} / (1-g)^a \qquad g < p < 1, \quad 0 < a < 1$$

.

Since the component reliabilities are assumed to be much more likely to be large, the values for α will be limited to the interval (0,1).

Further, assume that the lower limit on the reliability, γ, is a random variable that is more likely to take on smaller values. It is also defined by a beta density: Beta(1,α+1), $p(g \mid a) = (a+1)(1-g)^a$, where $0 < g < 1, 0 < a < 1$. Note that (see Figure 7.b) as α becomes smaller, the hyperprior $p(g \mid a)$ approaches a uniform distribution.

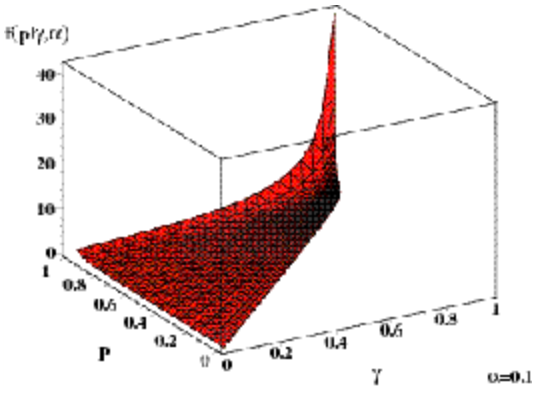

Figure 7.a Prior Family

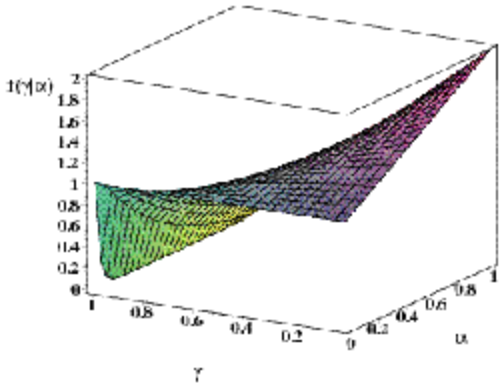

Figure 7.b Hyperprior Family

The directed acyclic graph depicting the relationships between the prior and hyperprior random variables and parameters is presented in Figure 8.

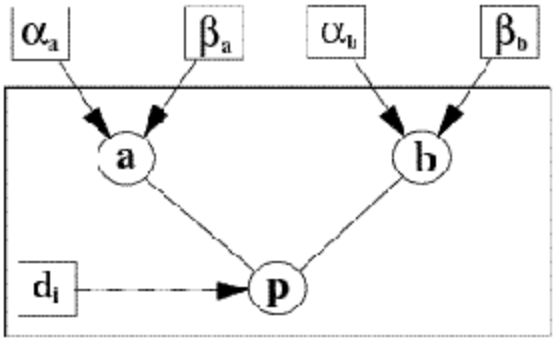

Figure 9. Component DAG for Example 2

Prior distribution – Example 2

In this situation, the first stage prior of the reliability will be assumed to follow as beta distribution defined on the full interval (0,1).

$$p(p \mid a,b) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} \quad 0 < p < 1, \quad a,b > 0$$

A common approach is to redefine the variable p via a logit transformation:

$$r = \log\left(\frac{p}{1-p}\right).$$

It is well known that if r is assumed to be a Gaussian random variable, then p is approximately beta. The second stage distributions for the first stage distribution parameters, m, s can then be conveniently constructed. For example, m can be assumed to be an improper uniform distribution defined across the real line and nl/s^2 can be assumed to follow a c_n^2 distribution where l and n are specified.

However, for the current situation, it will be assumed that the first stage prior remains described as a beta distribution and that the parameters, a and b, are each random variables further characterized in a second stage probability distribution.

The second stage distribution will be assumed to be a two parameter Weibull:

$$f(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^{a}\right] \qquad x > 0, a, b > 0$$

The DAG of this multi-stage analysis is depicted in Figure 9.

Component Data

Given that a set of n independent tests are performed on a component with s and f successes and failures are observed, then the data $\mathbf{d} = (n, s)$ can be incorporated into the posterior distribution function: $p(p \mid a,b,\mathbf{d})$.

Construction of Full Conditional Distributions (<u>without</u> system level test data)

For this example a first stage beta and second stage Weibull distributions will be assumed. The joint distribution of the data and the parameters of this model are then described by:

$$g(p,a,b,\mathbf{d}) = \prod_i g(p_i \mid a,b)g(a)g(b)$$

$$= \frac{p^{a+s-1}(1-p)^{b+f-1}}{B(a,b)} \frac{a_a a_b}{b_a b_b} \left(\frac{a}{b_a}\right)^{a_a-1} \left(\frac{b}{b_b}\right)^{a_b-1} \exp\left[-\left\{\left(\frac{a}{b_a}\right)^{a_a} + \left(\frac{b}{b_b}\right)^{a_b}\right\}\right] \qquad \text{(A)}$$

To construct the full conditional distribution $p(a \mid b)$ it is a simple matter to pick out those terms in (A) which involve the parameter a:

$$p(a \mid b) \propto a^{a_a-1} p^{a+s-1} \exp\left[-\left(\frac{a}{b_a}\right)^{a_a}\right].$$

Similarly, the full conditional distribution $p(a \mid b)$ can be found:

$$p(b \mid a) \propto b^{a_b-1}(1-p)^{b+f-1} \exp\left[-\left(\frac{b}{b_b}\right)^{a_b}\right]$$

Since these distributions cannot be sampled from directly, the Gibbs sampling approach can not be used directly. It is necessary to augment the sampling with a Metropolis-Hasting rejection step. Note that the full conditional distributions need to be known only up to a multiplicative constant since that constant will cancel in the reject step.

Simulation

The proposal distribution $q(\cdot)$ still needs to be specified. For the parameters *a* and *b*, the distribution will be assumed to be a Gaussian distribution with mean equal to the current value of the parameter and a standard deviation of twice the standard deviation of the associated marginal distribution. For p, the proposal distribution will be assumed to be $U_{0,1}$.

1. Assume initial values for $\{p_{(0)}, a_{(0)}, b_{(0)}\}$. These initial values are assumed to be the means of the respective marginal distributions.
2. Generate a random deviate for the parameter a:
   a. Generate a candidate random deviate, $a_{(i)}$, from the proposal distribution $y_{(i)} \sim N(a_{(i-1)}, 4s_a^2)$.
   b. Sample $u = U_{0,1}$

      If $u \le a(y_{(i)}, a_{(i-1)}, b_{(i-1)}, p_{(i-1)}) =$

      $$\min\left[\frac{p(a = y_{(i)} \mid b_{(i-1)}, p_{(i-1)})q(a = y_{(i)} \mid b_{(i-1)}, p_{(i-1)})}{p(a_{(i-1)} \mid b_{(i-1)}, p_{(i-1)})q(a_{(i-1)} \mid b_{(i-1)}, p_{(i-1)})}, 1\right] \Rightarrow a_{(i+1)} = y_{(i)}$$

      else $a_{(i+1)} = a_{(i)}$. The other parameters, *p* and *b*, are not updated.
3. Generate a random deviate for the parameter b:

a. Generate a candidate random deviate, $b_{(i)}$, from the proposal distribution $y_{(i)} \sim N(b_{(i-1)}, 4s_b^2)$.

b. Sample $u = U_{0,1}$

If $u \le a(y_{(i)}, a_{(i)}, b_{(i-1)}, p_{(i-1)}) =$

$$\min\left[\frac{p(b = y_{(i)} \mid a_{(i)}, p_{(i-1)})q(b = y_{(i)} \mid a_{(i)}, p_{(i-1)})}{p(b_{(i-1)} \mid a_{(i)}, p_{(i-1)})q(b_{(i-1)} \mid a_{(i)}, p_{(i-1)})}, 1\right] \Rightarrow b_{(i+1)} = y_{(i)}$$

else $b_{(i+1)} = b_{(i)}$. The other parameters, *p* and *a*, are not updated.

4. Generate a random deviate for the parameter p:

a. Generate a candidate random deviate, $p_{(i)}$, from the proposal distribution $y_{(i)} \sim U_{0,1}$.

b. Sample $u = U_{0,1}$

If $u \le a(y_{(i)}, a_{(i)}, b_{(i)}, p_{(i-1)}) =$

$$\min\left[\frac{p(p = y_{(i)} \mid a_{(i)}, b_{(i)})q(p = y_{(i)} \mid a_{(i)}, b_{(i)})}{p(p_{(i-1)} \mid a_{(i)}, b_{(i)})q(p_{(i-1)} \mid a_{(i)}, b_{(i)})}, 1\right] \Rightarrow p_{(i+1)} = y_{(i)}$$

else $p_{(i+1)} = p_{(i)}$. The other parameters, *a* and *b*, are not updated.

5. Steps 2-4 are repeated until a stationary sequence of samples $\mathbf{x}_{(i)}$ is available

Example

Let the conditional priors on *a* and *b* be Weibull distributions with means m_a and m_b respectively. These statistical characteristics have some intuitive relationship to the Bayesian concept of pseudo-successes and pseudo-failures. These parameters are variables input by the user depending on component historical information. For the following example, these parameters are assumed to be $m_a = 9.0$ and $m_b = 1.0$.

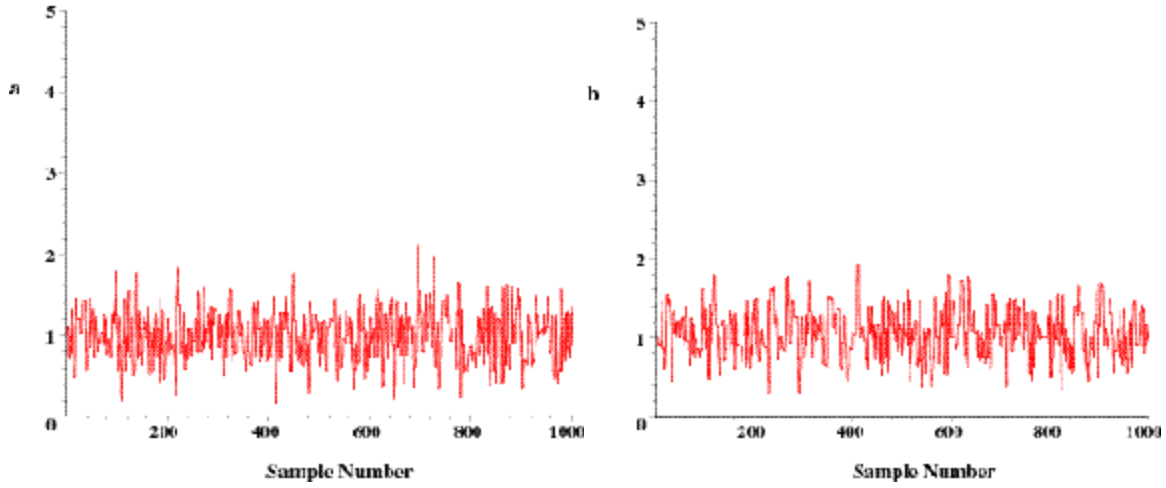

Figure 10.a-b Results for Component Reliability Example

Since there are two Weibull parameters, a second assumption is necessary. It will be assumed that the shape parameter, a, of both Weibull distributions is 3.5. From experience, the values of this parameter in the range [2.8-4.2] provide a consistent and stable response over a wide range of possible situations. For those situations where the component is considered to be very highly reliable, a shape parameter of 1.0 has also been found to be suitable.

These two conditions can be combined to estimate the location parameter of the Weibull distribution via the relationship: $b_i = m_i / \Gamma(1+1/a_i)$. The standard deviation of a and b must also be found and can be calculated using the expression: $s_i^2 = b^2\left[\Gamma\left(\frac{2}{a}+1\right) - \Gamma^2\left(\frac{1}{a}+1\right)\right]$. The standard deviation of both parameters is approximately 0.316. Using the parameter means as initial starting values, Figures 10.a and 10.b depict the results for the parameters a and b during the course of 1000 simulation runs. Figure 10.c is a plot of the observed component reliabilities $p(p \mid a,b)$ over the 1000 simulations runs and Figure 10.d is the associated frequency histogram.

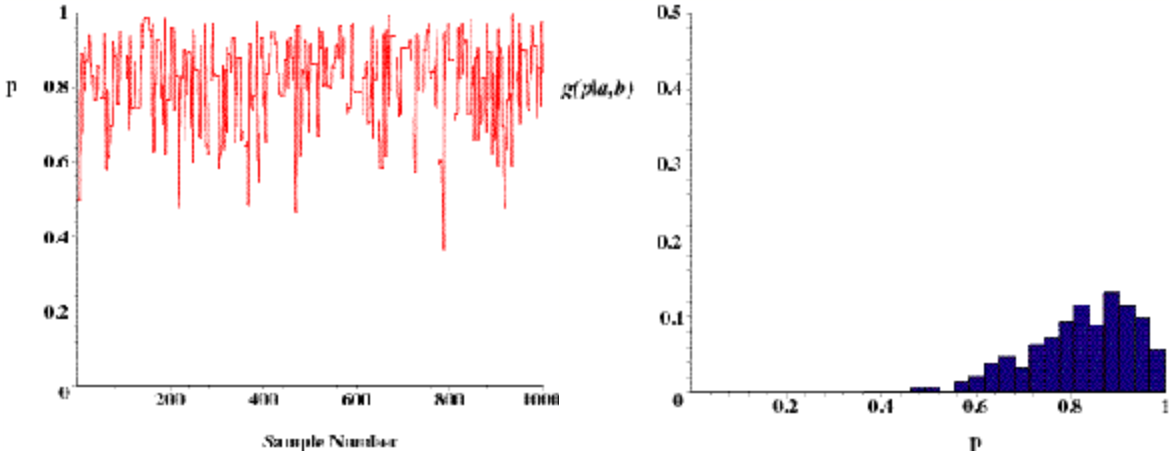

Figure 10.c-d Results for Component Reliability Example

From this analysis, a point estimate for the component reliability can be estimated:

$E[p] = \sum_{i=1}^{N} p(p_i \mid a,b) \Big/ N = 0.833$, along with the associated Bayes risk: $s_p = 0.1102$. Based on the simulation results, a 10% lower confidence limit of 0.68 and an upper 90% upper limit of 0.95 can be estimated for the component reliability.

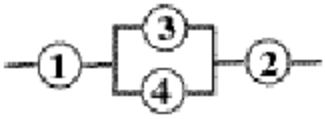

Figure 12. Simple System

Stationarity

A critical consideration when using any MCMC technique is the number of simulations required before stationarity is achieved. Given the above stage one and stage two distribution assumptions, the simulation process stabilizes very quickly. If instead of the mean values for a and b, arbitrary values had been chosen, the random process will still converge very fast as seen in Figures 11.a-b. In this case, initial values of 5.0 and 5.0 were used for initial values and the process becomes provides useful simulation results after only approximately 25 simulations.

System Level Analysis with Component Data

Independent Failure Modes

Before delving in the more complicated aspects of component interaction, a simple demonstration of the above approach will be provided. The system in question will be the simple 4 component system depicted in Figure 12.

An estimate of the system reliability will be constructed by combining simulated component reliability estimates through a coherent system function:

$$R_c = f(\mathbf{p}) = p_1 p_2 \left[1 - (1 - p_3)(1 - p_4)\right]$$

(R_c represents the reliability of the system based on component data, while R_s is defined as the system reliability based on system level data.) The vector of component

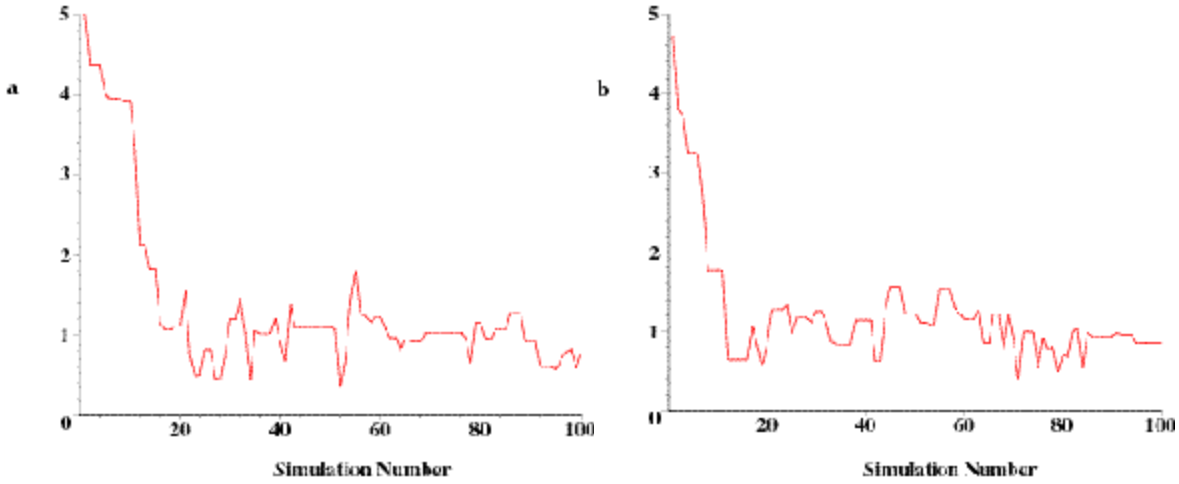

Figure 11.a-b Results Using Extreme Initial Values

reliabilities, **p**, is generated using the MCMC algorithm discussed in the previous discussion. Each vector of simulated component reliabilities is transformed using the system structure function and a system reliability estimate is collected. The result is an estimate of the posterior distribution of the system reliability: $p(R_c \mid q,\mathbf{d})$ given historical information and current failure data on each of the components. As with the component analyses, point and interval estimates of R_c can be computed. Also available is an estimate of the Bayes risk associated with using the historical component failure information.

The first stage prior for all for components is assumed to be Beta(a,b) and the second stage prior for all first stage parameters is assumed to be Weibull(α,β). The parameter values for the second stage priors are presented in Table 5. Given mean values for the parameters, and the assumption of a Weibull with shape parameter of 3.5, the location parameter for the Weibull can be easily found for each second stage prior. No historical data was assumed to be available on any of the components.

Component	μ_a	μ_b
p_1	1.60	0.18
p_2	9.10	1.01
p_3	18.19	4.27
p_4	3.94	0.92

Table 5. First and Second Stage Parameters

The **4C** software analysis package was used to complete the system level reliability analysis based on the component information. A histogram of the reliabilities for each of the components is presented in Figures 13.a-d. Figure 14.a depicts the system reliability under the situation where no historical data is available. The best estimate of the system reliability is 0.68 with a Bayes risk of 0.195 and an 80 percent confidence interval of [0.39, 0.90].

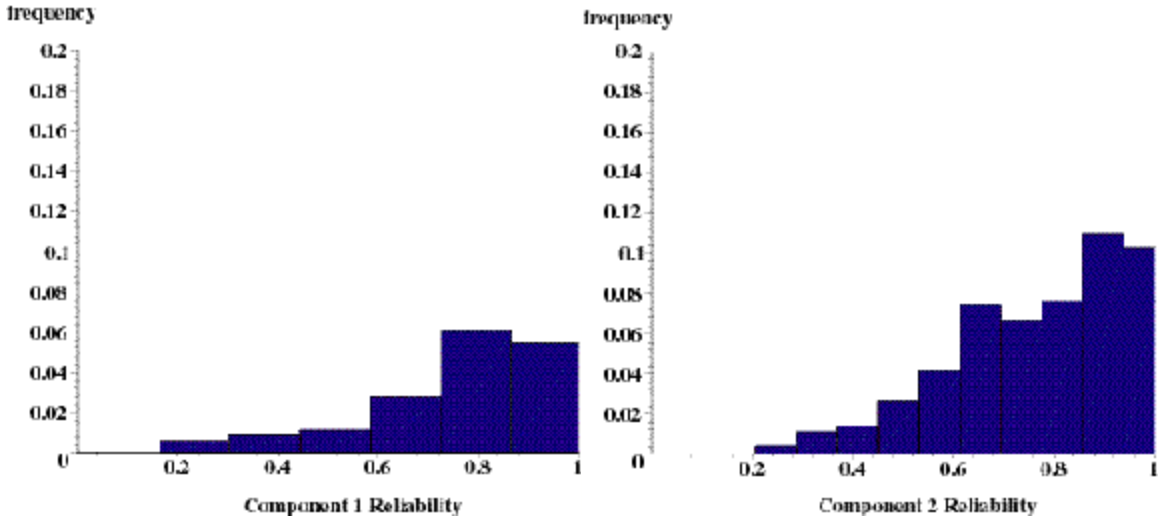

Figure 13.a-b Results for Components 1 and 2

In contrast, Figure 14.b depicts the situation where 10 tests were performed on component 1 and 9 successes observed. Combining this data with the prior beliefs about component one results in a new estimate of the system reliability of 0.74, a Bayes risk of 0.133, and an 80 percent confidence interval of [0.57, 0.90]. The change in the estimated reliability is evident in Figure 14.b, but equally important is the increased confidence in this estimate that results from the additional test data.

System and Component Data Aggregation

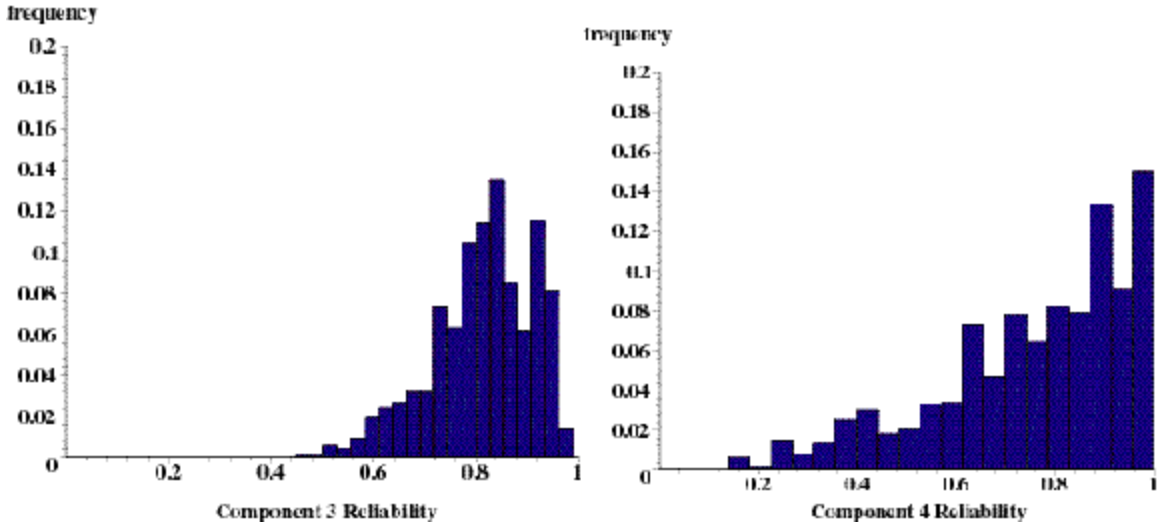

Figure 13.c-d Results for Components 3 and 4

It is often the case that testing is conducted at the system level and information regarding the failure or success of particular subsystems is not available or not collected. As noted previously in the discussion on data aggregation, care must be taken when interpreting system level test results and combining this information with component historical failure data.

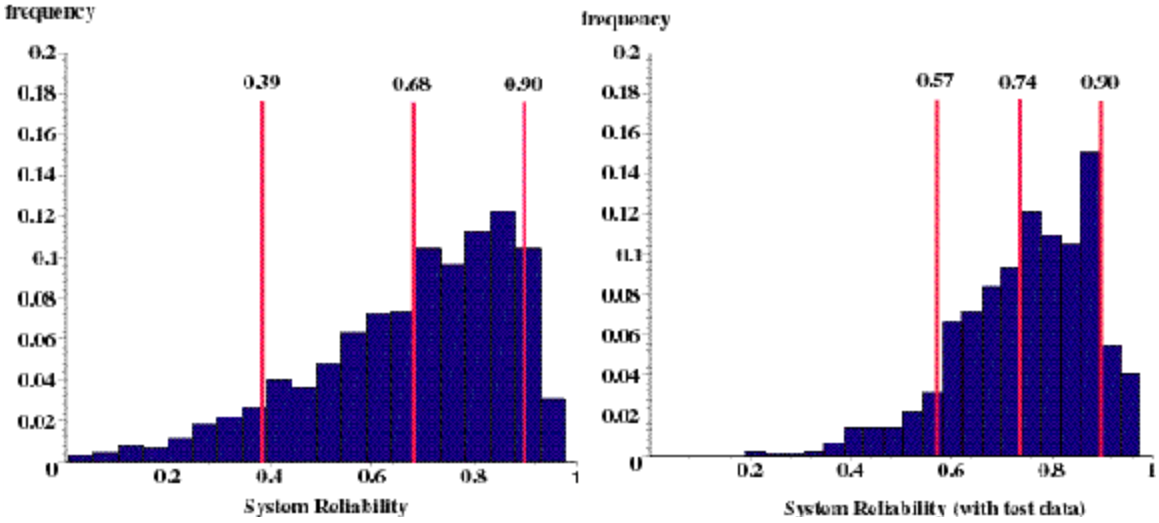

Figure 14.a-b Results for System Level Analyses

In this effort, the system level posterior distribution function that results from combining component test information will be used as a prior distribution for the system level reliability analysis. Based on the previous discussion regarding the binomial testing of components:

$$g(p_s \mid s) = KR_s^{\,S}(1-R_s)^F g(p_s)$$
$$\propto R_s^{\,S}(1-R_s)^F p(p_c \mid \mathrm{q}, \mathbf{d})$$

where S and F are the system level successes and failures respectively. The posterior distribution function defined as a function of a set of parameters and data, now becomes the prior distribution for the system reliability: $p(p_c \mid \mathrm{q}, \mathbf{d}) \Rightarrow g(p_s)$.

This is generally available as a histogram of M intervals constructed from the results of an analysis such as an MCMC simulation described above. To construct the histogram of the system reliability based on both component and system data:

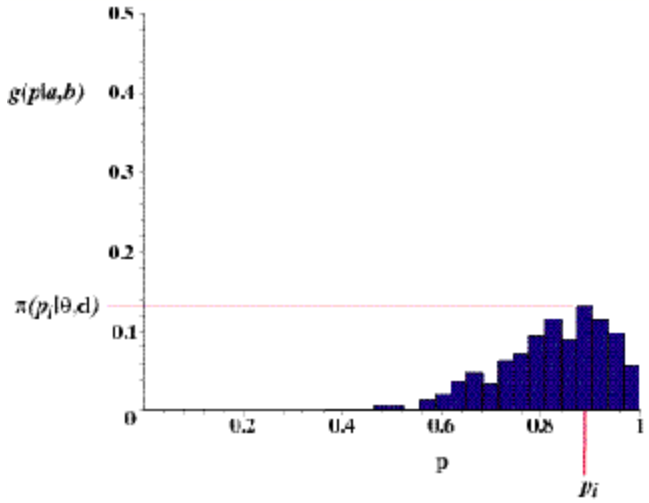

Figure 15. Elements of Component Histogram

$$G[p_i \mid S,F] = \frac{1}{B(S,F)} p_i^{\,S}(1-p_i)^F p(p_i \mid \mathrm{q}, \mathbf{d}) \quad i = 1,\ldots,M$$

The nth moment of the system reliability is then given by:

$$E[p_S^n] = \frac{1}{B(S,F)} \sum_{i=1}^{M} p_i^{\,n} \left[p_i^{\,S}(1-p_i)^F p(p_i \mid \mathrm{q}, \mathbf{d}) \right]$$

and a point estimate and associated Bayes risk can be easily constructed (see Figure 15 for the definition of $p(p_i \mid \mathrm{q}, \mathbf{d})$).

Example

The **4C** software was again used to incorporate system level failure data into a reliability analysis of the simple system depicted in Figure 12. Results from a total of nine different test examples are presented including the base case where no component and no system test data are available. Significant differences between cases are highlighted in red. Figure 16 depicts a typical prior probability density function for the reliability of a component for the base case. Typical sample prior distributions for the parameters a and b are presented in Figure 17. The resulting system

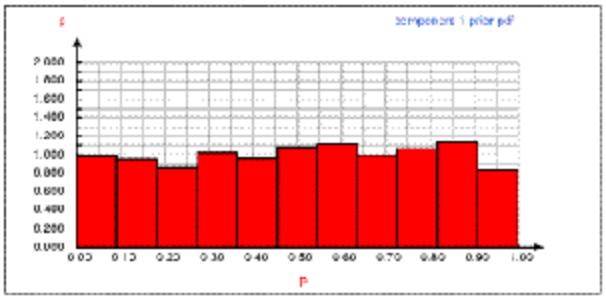

Figure 16. Typical Component Reliability Prior

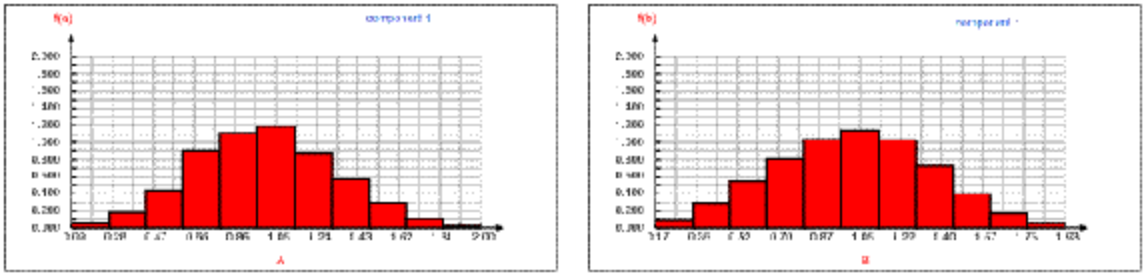

Figure 17. Typical Parameter Prior Density Functions

reliability probability density function is depicted in Figure 18.

Cases 2-4 illustrate how the assumption of prior information for each of the major types of components impacts the final system reliability estimate and the uncertainty in the reliability. (Note that the confidence in the reliability estimate is presented in terms of the variance of the reliability estimate or equivalently, the Bayes risk. As the variance in the reliability estimate decreases, the confidence in the estimate increases.)

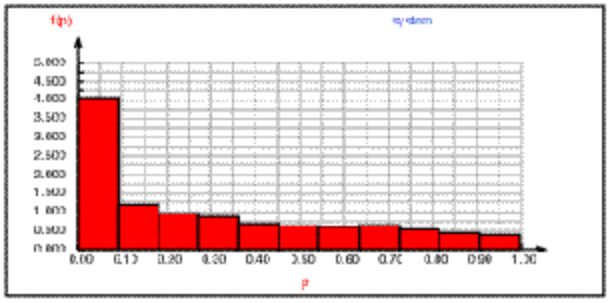

Figure 18. Typical System Reliability

The consequences of data aggregation is investigated in cases 5-9. Cases 5-7 illustrate the situation where a total of 10 system level tests were conducted and failures were isolated to 2 components. Case 5 involves the situation where all data is aggregated to the system level for reliability estimation. Two different scenarios for failure observation are investigated in cases 6 and 7: first, failures are observed on each of the parallel components, see Figure 19 for typical component reliabilities for case 6. Secondly, the failures were assumed to have all occurred on a single series unit. The variation in the reliability for these two cases in comparison to the system level results (case 5), highlights the need for extreme caution when using system level data for a reliability analysis. In particular, note the difference in the probability density functions for the system reliability illustrated in Figure 22. Finally, in case 8 test data is available only at the system level; due to instrumentation

Component	**1**				**2**				**3**				**4**				**System**		System Level Results	
	data		prior		data		prior		data		prior		data		prior		data			
Case	S	F	S	F	S	F	S	F	S	F	S	F	S	F	S	F	S	F	μ	σ
1	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	0.300	0.295
2	0	0	**5**	**5**	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0.216	0.200
3	0	0	1	1	0	0	1	1	0	0	**5**	**5**	0	0	1	1	0	0	0.291	0.281
4	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	**5**	**5**	0.481	0.143
5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	**8**	**2**	0.733	0.121
6	**10**	**0**	1	1	**10**	**0**	1	1	**9**	**1**	1	1	**9**	**1**	1	1	0	0	0.744	0.158
7	**8**	**2**	1	1	**10**	**0**	1	1	**10**	**0**	1	1	**10**	**0**	1	1	0	0	0.632	0.147
8	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	**3**	**0**	0.761	0.174
9	**3**	**0**	1	1	**3**	**0**	1	1	**0**	**3**	1	1	**3**	**0**	1	1	0	0	0.502	0.268

Table 6. Results of 9 Sample Cases

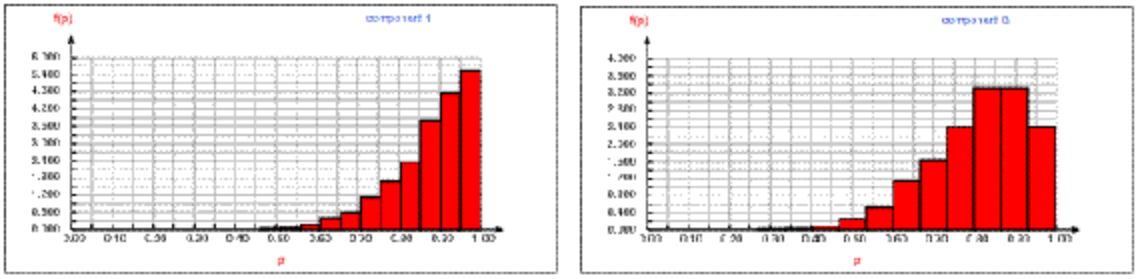

Figure 19. Typical Component Reliabilities – Case 6

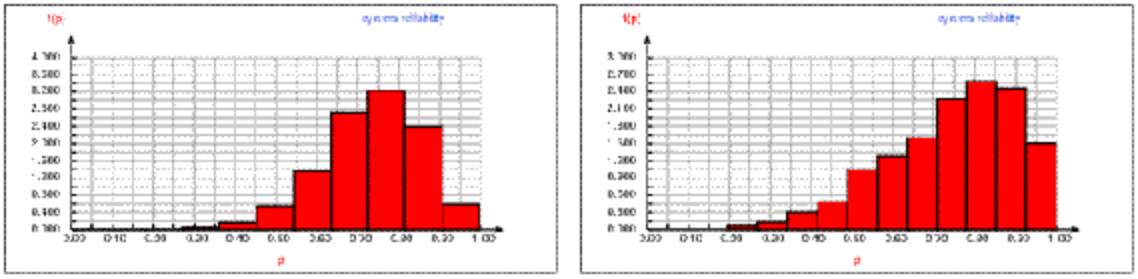

Figure 19. Typical System Reliabilities – Case 5 and Case 6

costs it was not possible to collect component failure information during the flight test. This scenario is very typical in operational testing of air-to-ground missiles. In reality, there has been 3 component failures, all involving a component that is part of a redundant segment of the system. The reliability estimates that result from these two different cases is vastly different (0.761 versus 0.502).

Correlated System Failure Modes

A complicating factor in system reliability analyses is the possibility that various components of $f[p(p_1,p_2,\ldots,p_n \mid \mathbf{a,b,d})]$ may be statistically correlated. In more general situations a system may have a number of possible failure modes that are statistically related to each other through the physics of system operation. To address this issue, a little known family of multivariate distributions will be introduced and subsequently applied.

Bivariate Sarmanov Distribution

In 1966 Sarmanov developed a bivariate family of distributions which have the capability of characterizing highly correlated random variables. Lee (1996) rediscovered this work and suggested its possible use in the area of Bayesian analysis; the following discussion draws heavily from that paper.

For the bivariate case, define the parameter vector $\mathrm{q}_i, \quad i=1,2$. Let $p(p_i \mid \mathrm{q})$ be univariate probability density functions and let $j(x_i)$ be bounded, nonconstant functions where $\int j(p_i) f(p_i) dp_i = 0$. Then the function:

$$p(p_1,p_2 \mid \mathrm{q}_1,\mathrm{q}_2,w_{12}) = p(p_1 \mid \mathrm{q}_1)p(p_2 \mid \mathrm{q}_2)\{1+w_{12}j(p_1)j(p_2)\}$$

is a bivariate joint density function with $1+w_{12}j(p_1)j(p_2) \geq 0$, for all p_i. Note that a similarity measure $w_{12}=0$ implies that p_1 and p_2 are statistically independent. The

correlation coefficient of p_1 and p_2 can be derived: $r = \frac{w_{12}\eta_1\eta_2}{\sigma_1\sigma_2}$ where $\eta_i = \int_{-\infty}^{+\infty} p_i \phi(p_i) p(p_i \mid q)\, dp_i$ and σ_i is the standard deviation of p_i.. The range of the similarity measure w_{12} can also be shown to be bounded:

$$\max\left(\frac{-1}{m_1 m_2}, \frac{-1}{(1-m_1)(1-m_2)}\right) \le w_{12} \le \min\left(\frac{-1}{m_1(1-m_2)}, \frac{-1}{m_2(1-m_1)}\right)$$

If $p(p_i \mid q)$ is a beta distribution function and the mixing function is defined as $\phi_i(x) = p_i - E[p_i]$, then the correlation coefficient simplifies to: $r = w_{12}\sigma_1\sigma_2$ and the bounds on the similarity measure simplify to:

$$\frac{-(a_1+b_1)(a_2+b_2)}{\max(a_1 a_2, b_1 b_2)} \le w_{12} \le \frac{(a_1+b_1)(a_2+b_2)}{\max(a_1 b_2, a_2 b_1)}$$

Multivariate Sarmanov Distribution

Lee (1996) also generalized the similarity relationship above to *n* dimensions. Define a relationship parameter

$$R_{123}(p_1,p_2,p_3) = w_{12}\phi(p_1)\phi(p_2) + w_{13}\phi(p_1)\phi(p_3) + w_{23}\phi(p_2)\phi(p_3) + w_{123}\phi(p_1)\phi(p_2)\phi(p_3)$$

where: $w_{123} = \frac{E[(p_1-m_1)(p_2-m_2)(p_3-m_3)]}{\sigma_1^2\sigma_2^2\sigma_3^2}$ and $m_i = E[p_i]$. Then

$$p(p_1,p_2,p_3 \mid q_1q_2q_3w) = \prod_{i=1}^{3} p(p_i \mid q_1q_2q_3)\{1 + R_{123}(p_1,p_2,p_3)\}.$$

The expression can be generalized:

$$p(p_1,p_2,\ldots,p_n \mid q,w) = \prod_{i=1}^{n} p(p_i \mid q)\{1 + R_{12\ldots n}(p_1,p_2,\ldots,p_n)\}$$

Sampling Approach

Generation of the set of system component reliabilities:

$$p(p_j \mid p_1,\ldots,p_{-j},\ldots,p_n \mid q,w) = p(p_j \mid q)\left\{\frac{1 + R_{12\ldots n}(p_1,p_2,\ldots,p_n)}{1 + R_{12\ldots(-j)n}(p_1,p_2,\ldots,p_{-j},\ldots,p_n)}\right\}$$

can be accomplished easily using the Metropolis-Hastings rejection sampling scheme. Note that $p(p_j \mid q)$ is a beta distribution and that since $|R| \le 1$:

$$\left\{\frac{1 + R_{12\ldots n}(p_1,p_2,\ldots,p_n)}{1 + R_{12\ldots(-j)n}(p_1,p_2,\ldots,p_{-j},\ldots,p_n)}\right\} \le 2 = M$$

A sample taken and accepted if: $M \cdot U_{0,1} \le \left\{ \dfrac{1+ R_{12..n}(p_1, p_2, \ldots, p_n)}{1+ R_{12..(-j)n}(p_1, p_2, \ldots, p_{-j}, \ldots, p_n)} \right\}$.

Expert Elicitation to Construct Similarity Relationships

Recall that the similarity measure can be defined in terms of the correlation coefficient: $w_{12} = r_{12} / s_1 s_2$. Some algebra results in the expression:

$$w_{12} = \left(E[p_1 \mid p_2] - E[p_1]\right) \Big/ s_1^2 \left[p_2 - E[p_1]\right]$$
$$= \Omega_{12} / \left[p_2 - m_1\right]$$

Obviously the best approach to constructing the similarity measure is via an estimate of the correlation function based on actual test data. However, if specific data regarding the correlation between component reliability or failure modes is not available, the parameter $\Omega_{12} = \left(E[p_1 \mid p_2] - m_1\right) \Big/ s_1^2$ can be constructed via expert elicitation of $E[p_1 \mid p_2]$; the expected values of p_1 for given values of p_2.

These similarity measures might also be the result of expert elicitation using those techniques being developed at LANL and implemented in the PREDICT software package.

Discussion

The area of correlated failure modes and the merging of data from similar systems is a very fruitful area of discussion. There are many examples and situations where such an approach would be critical to making the most efficient use of the available test data. For this reason, discussion of specific examples is presented under a separate technical report to be released in the near future.

Summary

This report has provided a mathematical foundation for an approach to bring together data from a variety of sources and provided an approach to combine this data in a logical fashion. The application of this approach will result in an increased confidence in the reliability assessment of the nuclear weapons stockpile.

The benefits of using a Bayesian approach were very briefly reviewed and two alternatives to Bayesian analysis were discussed. The hierarchical Bayes approach was suggested as having more widespread applicability and also reduced the sensitivity of the analysis to the assumptions regarding the statistical structure of the prior information.

The fundamental aspects of Monte Carlo Markov Chain (MCMC) were reviewed including a brief discussion of the methods associated with Metropolis-Hastings and Gibbs sampling. A methodology for the application of MCMC methods to component reliability was outlined and applied to a simple example. The approach was extended to allow the assessment of system reliability using both component prior information and component test data.

Finally, an approach to aggregating system level test data and component test data was developed and demonstrated through a sequence of test cases.

The discussion and examples presented in this report have been limited to those systems composed of discrete components, i.e. those that can be characterized as operating or not operating. However, there is nothing unique about the underlying principles and the extension to continuous subsystem/systems is straightforward. In addition, an approach for considering the possibility of correlated failure modes was also developed. Further discussion and application of this approach is reserved for a future research report.

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The references presented here are by no means exhaustive. There are simply those citations that were found to be particularly useful in the course of this investigation and are provided for those readers interested in additional detail. Topic area include: General Bayes – GB, Empirical Bayes – EB, Hierarchical Bayes – HB, Markov Chain Monte Carlo – MCMC, Bayesian Test Planning – TP.

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Appendix A. Function Definitions

Name	Density Function	Mean	Variance
Beta(a,b)	$\frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} \quad 0 < p < 1, \quad a,b > 0$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$
Truncated *Beta(a,b,p_0,p_1)*	$\frac{1}{B(a,b)(p_1-p_0)^{a-1}}(p-p_0)^{a-1}(p_1-p)^{b-1}$ $0 \le p_0 < p \le p_1 < 1, \quad a,b > 0$		
Weibull(a,b)	$\frac{a}{b}\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^{a}\right] \qquad x > 0, a,b > 0$	$b\Gamma\left(\frac{1}{a}+1\right)$	$b^2\left[\Gamma\left(\frac{2}{a}+1\right)-\Gamma^2\left(\frac{1}{a}+1\right)\right]$
InvGam(a,b)	$\frac{b^a}{\Gamma(a)}\left(\frac{1}{x}\right)^{a+1} \exp\left(-\frac{b}{x}\right), \quad x,a,b > 0$	$\frac{b}{a-1}, \quad a > 1$	$\frac{b^2}{(a-1)^2(a-2)}, \quad a > 2$
B(a,b)	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, \quad \Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx,$ or if n is integer $> 0, \ \Gamma(n) = (n-1)$		

Appendix B. Monte Carlo Sampling

Monte Carlo Sampling

For each random variable in a problem, a random deviate is chosen from the sampling probability density function. Using these values a vector of random inputs is constructed, $\mathbf{X}^i$. This vector of random deviates is used as parameter settings in system model and a system response, *G*, is calculated. A series of *n* sample vectors are generated, $\{\mathbf{X}^1,\ldots,\mathbf{X}^n\}$, and associated system response calculated until sufficient samples have been generated such that the analyst is confident that the uncertainty in the system response has been adequately characterized.

The following discussion is extremely brief; there are scores of books dedicated to Monte Carlo sampling techniques. This section is included primarily for completeness and to provide a basis for discussing variations on the Monte Carlo theme.

Random Number Generation

An essential element in any Monte Carlo scheme is the generation of random numbers. Knuth (1981) provides an excellent discussion, and the following is a brief look at only the more common approaches. It will be assumed that, as a minimum, the generation of uniformly distributed random deviates from the interval [0,1] is available. Notationally, these random deviates will be denoted: $U_{0,1}$. A random deviate from a Gaussian density function with mean m and variance s^2 will be denoted: $N(m,s^2)$.

There are three fundamental methods for generating random numbers: inversion, transformation and rejection. Only a brief introduction to these methods will be presented; the reader is again referred to the many books on simulation, e.g. Ripley (1987) or Rubenstein (1981).

Inversion

Inversion is the most straightforward method of generating random numbers from a particular density function. This technique is based on the availability of the inverse cumulative density function for the density function of interest. For some probability density functions this is straightforward, as in the case of the exponential density. In other cases, such as the Gaussian probability density function, this inverse does not exist.

Assume that the cumulative density function of the random variable is known, $y = P\{X \le x\} = F(x)$. Since y can take on any value in the interval [0,1], it is a simple matter to generate a random value y_i from $U_{0,1}$ and then solve: $x_i = F^{-1}(U_{0,1})$.

Example

Given a random variable x with probability density and cumulative density functions:

$$f(x) = l \exp\{-l x\}$$
$$F(x) = 1 - \exp\{-lx\} \qquad l, x \ge 0$$

The inverse of the cumulative function is easily found:

Since (1-y) is distributed $U_{0,1}$, y is also distributed $U_{0,1}$. The above expression can be simplified somewhat and a sequence of random variables from an exponential distribution can be generated by repeated application of:

$$x_i = F^{-1}(U_{0,1}) = -\frac{1}{l}\ln(U_{0,1})$$

Transformation

While the inversion method is extremely straightforward, it depends on the ease with which the inverse cumulative density function can be computed either analytically or numerically. In those cases where the inverse is not easily available, an alternative method for deviate generation is the transformation method. The basic idea behind the transformation method is to develop a function, which when applied to uniformly distributed random deviates, transforms these random variables into deviates from the desired density function. The inverse method described above is, as might be suspected, a unique case of the transformation method.

Assume a general two-dimensional case, where it is desired to generate a random number from the density function, $f(x_1, x_2)$. This density function can be described as a result of the transformation:

$$f(x_1, x_2) = g(y_1, y_2)\left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right|$$

where $g(y_1, y_2)$ is some function of the new variables y_1, y_2 and $\left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right|$ is the Jacobian of the transformation.

Example

The most famous application of the above approach is the Box-Muller method for generating random deviates from the Gaussian probability density function. Let y_1, y_2 be independently distributed with density functions, $U_{0,1}$. Define:

$$x_1 = \sqrt{-2\ln(y_1)}\cos(2\pi y_2)$$
$$x_2 = \sqrt{-2\ln(y_1)}\sin(2\pi y_2)$$

therefore:

$$y_1 = \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right]$$
$$y_2 = \frac{1}{2\pi}\arctan\left[\frac{x_2}{x_1}\right]$$

The Jacobian is then:

$$\left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = \left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right)\right]\left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2}\right)\right]$$

which is just the product of two independent, standardized, Gaussian distributed random variables. Therefore, inputting two, independent random variables y_1, y_2 from $U_{0,1}$, two independent, Gaussian distributed random variables, $N_{0,1}$ can be generated.

Acceptance-Rejection Sampling

Assume that we wish to generate a sample from a probability density function $f(x)$, but for computational reasons the inversion or transformation methods are not practical. However, assume that a straightforward method exists for generating samples from a second, candidate probability density function $g(x)$ where $f(x) < Mg(x)$ and M is any scalar multiplier. Then to generate a sample from $f(x)$ the following procedure can be used (see Figure B.1):

1. generate a random deviate, *x*, from $g(x)$
2. sample $u = U_{0,1}$
3. Accept *x* as sample if $u < \frac{f(x)}{Mg(x)}$
   else return to Step 1

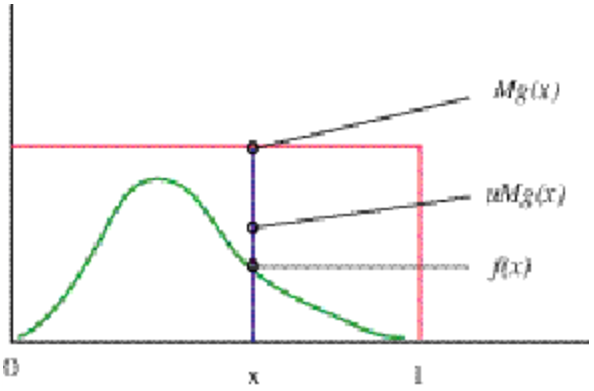

Figure B.1. Rejection Sampling

A small value of *M* is desired since the number of samples that must be generated until an acceptable sample is generated follows a geometric distribution with mean *M*. The efficiency of the algorithm is also sensitive to the choice of *g(x)*, i.e. the closer *g(x)* bounds *f(x)* the more efficient the sampling procedure.

Example

Given a random variable *x* with probability density function (Beta):

$$f(x) = x^{a-1}(1-x)^{b-1} \qquad a, b \geq 1$$

The bounding distribution, *g(x)* is distributed $U_{0,1}$. The multiplier is then defined:

$$M = \frac{(a-1)^{a-1}(b-1)^{b-1}}{(a+b-2)^{a+b-2}}$$

The Metropolis-Hastings algorithm discussed in the main body is a variation of the above rejection sampling method. However, rather than accepting/rejecting *x* as independent samples from a desired distribution, the generated sample must depend on a previous value. The resulting set of samples constitutes a Markov process.

Distribution

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